Robust Optimization & Network Design
Lecture 7

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Lehrstuhl II für Mathematik
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Outline

Recall: In many problems, all decisions have to be made in advance.

Discrete Scenarios

- limited number of scenario vectors
- solution should be valid for all scenarios

Γ Scenarios (Bertsimas & Sim 03/04)

- demand $d^k \in [\bar{d}^k, \bar{d}^k + \hat{d}^k]$ with nominal demand $\bar{d}^k$ and deviation $\hat{d}^k$
- due to statistical multiplexing only few simultaneous peaks
- assume at most Γ peaks at same time
- solution should be valid for all scenarios

In both cases: optimize worst-case

Drawback: “almost always” good solutions might be infeasible

Two-Stage RO: some decisions are only taken at 2nd stage

Recoverable robustness: repair 1st stage decisions

- uncertainty as two-stage process:

  1st stage: a-priori decision

  2nd stage: recovery:
  limited change of first-stage decision
  after realization of uncertainty is known

- optimize worst-case w. r. t. recovery

In this lecture: Recoverable Robust Knapsack problem (RRKP) with

- Discrete Scenarios[1]
- Γ Scenarios[2]

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[1] C. Büsing, A. M. C. A. Koster, and M. Kutschka. Recoverable robust knapsacks: the discrete scenario case. *Optimization Letters*, 5(3):379–392, 2011

[2] C. Büsing, A. M. C. A. Koster, and M. Kutschka. Recoverable robust knapsacks: *gamma*-scenarios. In *Proceedings of INOC 2011*, volume 6701 of *Lecture Notes on Computer Science*, pages 583–588, 2011

(k,ℓ)-RRKP with Discrete Scenarios

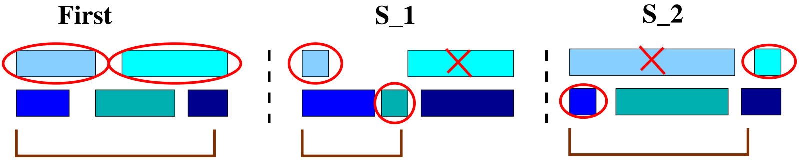
Given
- items $N = \{1, \ldots, n\}$,
- first stage: profits p^0, weight w^0, capacity c^0,
- scenarios $S \in \mathcal{S}_D$ with profits p^S, weight w^S, capacity c^S,
- recovery set $\mathcal{X}(X)$: delete $\leq k$ items, add $\leq \ell$ items

Find
- subset $X \subseteq N$

Such that
- $w^0(X) \leq c^0$,
- for all $S \in \mathcal{S}_D$ there exists $X^S \in \mathcal{X}(X)$ with $w^S(X^S) \leq c^S$,
- total profit

$$p_T(X) = p^0(X) + \min_{S \in \mathcal{S}_D} \max_{X^S} p^S(X^S)$$

is maximized.



RRKP with Discrete Scenarios - ILP

$$
\begin{aligned}
\max \sum_{i \in N} p_i^0 x_i + \omega & & & \\
s.t. \sum_{i \in N} w_i^0 x_i & \leq c^0 & & \\
\sum_{i \in N} w_i^S x_i^S & \leq c^S & & \forall S \in \mathcal{S}_D \\
x_i - x_i^S - y_i^S & \leq 0 & & \forall S \in \mathcal{S}_D, i \in N \\
\sum_{i=1}^{n} y_i^S & \leq k & & \forall S \in \mathcal{S}_D \\
x_i^S - x_i - z_i^S & \leq 0 & & \forall S \in \mathcal{S}_D, i \in N \\
\sum_{i=1}^{n} z_i^S & \leq \ell & & \forall S \in \mathcal{S}_D \\
\omega - \sum_{i=1}^{n} p_i^S x_i^S & \leq 0 & & \forall S \in \mathcal{S}_D \\
x_i, x_i^S, y_i^S, z_i^S \in \{0,1\} & & &
\end{aligned}
$$

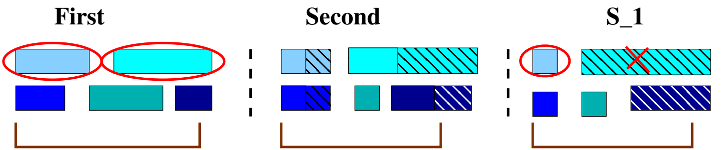
first stage
second stage
removal of $\leq k$ items
addition of $\leq \ell$ items

k-RRKP with Γ Scenarios

Given
- Items $N = \{1, \ldots, n\}$,
- first stage: profits p^0, weights w^0, capacity c^0,
- Γ-scenarios: weights $[\bar{w}, \bar{w} + \hat{w}]$, capacity c, $\Gamma \in \mathbb{N}$,
- recovery set $\mathcal{X}(X)$: delete $\leq k$ items from $X \subseteq N$

Find
- subset $X \subseteq N$,

Such that
- $w^0(X) \leq c^0$,
- for all $S \in \mathcal{S}_\Gamma$ there exists $X^S \in \mathcal{X}(X)$ with $w^S(X^S) \leq c$,
- total profit $p^0(X)$ is maximized



RRKP with Γ scenarios - MP

Mathematical Programming formulation:

$$\max \sum_{i \in N} p_i^0 x_i$$

$$s.t. \sum_{i \in N} w_i^0 x_i \leq c^0$$

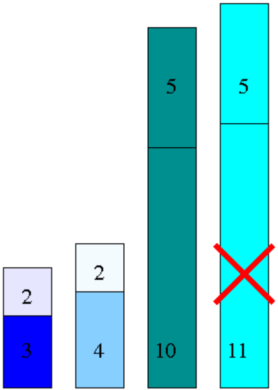
$$\sum_{i \in N} \bar{w}_i x_i + \max_{\substack{X \subseteq N \\ |X| \leq \Gamma}} \Big(\sum_{i \in X} \hat{w}_i x_i - \max_{\substack{Y \subseteq N \\ |Y| \leq k}} \Big(\sum_{i \in Y} \bar{w}_i x_i + \sum_{i \in X \cap Y} \hat{w}_i x_i \Big)\Big) \leq c$$

$$x_i \in \{0, 1\}$$

Question: Compact Linear reformulation?
Answer: LP duality and enumeration of solution values!

Max Weight Set Problem

Example:



Given	Items N (1st stage solution) weight bounds $[\bar{w}, \bar{w} + \hat{w}]$ parameters Γ (robustness) and k (recovery)
Find	Items $Z \subseteq N$, $\lvert Z \rvert \leq \Gamma$
Such that	total weight of recovery set is maximized (i.e., set after recovery)

$\Gamma = 2,\ k = 1,\ \text{Opt} = 21$

- Choice of Z for $\Gamma = 2$ does not include choice for $\Gamma = 1$!
- Reformulation by LP duality!

Let $U := \{0\} \cup \{\bar{w}_i : i \in N\} \cup \{\bar{w}_i + \hat{w}_i : i \in N\}$
Then, a compact reformulation is:

$$
\begin{aligned}
\max \sum_{i\in N} & p_i^0 x_i \\
s.t. \sum_{i\in N} & w_i^0 x_i && \leq c^0 \\
& \sum_{\substack{i\in N:\\ \bar{w}_i < u}} \bar{w}_i x_i + \sum_{\substack{i\in N:\\ \bar{w}_i \geq u}} u x_i + \Gamma \xi^u + \sum_{i\in N} \theta_i^u && \leq c + ku && \forall u \in U \\
& \min\{-\bar{w}_i + u, \hat{w}_i\} x_i - \xi^u - \theta_i^u && \leq 0 && \forall u \in U \\
& \xi^u, \theta_i^u \geq 0 && && \forall u \in U, i \in N \\
& x_i \in \{0, 1\}
\end{aligned}
$$

Resulting compact model contains $\mathcal{O}(n^2)$ variables and constraints

Complexity of Robust Knapsack

Theorem 1 (Karp 72, Bellman 57)

The knapsack problem is weakly **NP**-*hard, i.e., it can be solved in* $O(nc)$ *time.*

Theorem 2 (Yu 96, Kalai & Vanderpooten 06)

The robust knapsack problem with bounded number of scenarios can be solved in pseudo-polynomial time.

Theorem 3 (Yu 96, Aissi et al. 07)

The robust knapsack problem with discrete scenarios is strongly **NP**-*hard and not approximable, unless* $\mathbf{P} = \mathbf{NP}$.

Theorem 4 (Bertsimas & Sim 03/04, Klopfenstein & Nace 08)

The Γ-*robust knapsack problem can be solved in pseudo-polynomial time.*

Complexity of RRKP with Discrete Scenarios

Theorem 5

The (k, ℓ)-rrKP is strongly **NP***-hard for unbounded sets of discrete scenarios even if either $p^0 = 0$ or $p^S = 0$ for all $S \in \mathcal{S}_D$ holds.*

Reductions from 3SAT.

Corollary 6

The (k, ℓ)-rrKP cannot be approximated within $\frac{\ell+1}{\ell}$, unless **P** $=$ **NP***. In particular, for $\ell = 0$, the problem cannot be approximated.*

Theorem 7

The (k, ℓ)-rrKP can be solved in pseudo-polynomial time for a bounded number of scenarios.

Generalization of dynamic programming for robust knapsack (Yu, 1996).

Theorem 8

The RRKP with Γ scenarios is at least weakly **NP***-hard.*

Open Problem

Is the RRKP with Γ scenarios strongly **NP**-hard or does there exist a pseudo-polynomial time algorithm?

Projection of Discrete Scenarios

Let $p^S \equiv 0$ for all $S \in \mathcal{S}_D$.
$\Rightarrow$ The number ℓ of items added do not play a role

Definition 9 (RRK Polyhedron)

$$\mathcal{K}_D(k) := \operatorname{conv}\left\{ x \in \{0,1\}^n : \sum_{i \in N} w_i^0 x_i \leq c^0 \text{ and } \min_{\substack{T \subseteq N \\ |T| \leq k}} \sum_{i \in N \setminus T} w_i^S x_i \leq c^S \ \forall S \in \mathcal{S}_D \right\}$$

Projection on original variables

If $p^S \equiv 0$, the (k, ℓ)-RRKP with discrete scenarios can be formulated as

$$\max \left\{ \sum_{i \in N} p_i^0 x_i : x \in \mathcal{K}_D(k) \right\}$$

Definition 10 (Γ-RRK Polyhedron)

$$\mathcal{K}_\Gamma(k) := \text{conv}\left\{ x \in \{0,1\}^n : \sum_{i \in N} w_i^0 x_i \leq c^0 \text{ and } \min_{\substack{T \subseteq N \\ |T| \leq k}} \sum_{i \in N \setminus T} w_i^S x_i \leq c \ \forall S \in \mathcal{S}_\Gamma \right\}$$

Projection on original variables

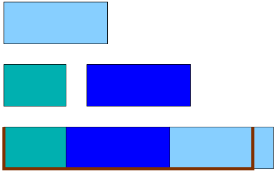
The Γ-RRKP can be formulated as

$$\max \left\{ \sum_{i \in N} p_i^0 x_i : x \in \mathcal{K}_\Gamma(k) \right\}$$

How do $\mathcal{K}_D$ and $\mathcal{K}_\Gamma$ look like?

Non-robust Knapsack Polytope:

$$\mathcal{K} := \text{conv}\{x \in \{0,1\}^n \ : \ \sum_{i=1}^{n} w_i^0 x_i \leq c^0\}$$



- Cover C : $\sum_{i \in C} w_i^0 \geq c^0 + 1$
- Cover inequality :

$$\sum_{i \in C} x_i \leq |C| - 1$$

Non-robust cover: If $\sum_{i \in C} w_i^0 \geq c^0 + 1$, then $\sum_{i \in C} x_i \leq |C| - 1$.

Definition 11

A set $C \subseteq N$ is called an *rrKP cover* if

first stage cover: $w^0(C) \geq c^0 + 1$ or

scenario cover: $w^S(C) - w^S(\max, C, k) \geq c^S + 1$,

where $w^S(\max, C, k) := \max_{\substack{B \subseteq C \\ |B| \leq k}} w^S(B)$.

Theorem 12

Given a rrKP cover C, the rrKP cover inequality

$$\sum_{i \in C} x_i \leq |C| - 1$$

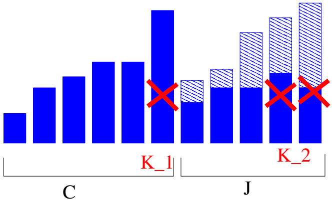
is valid for $\mathcal{K}_D$.

$C \subseteq N$ nominal items ($w_i = \bar{w}_i$)

$J \subseteq N$ peak items ($w_i = \bar{w}_i + \hat{w}_i$)

$K_1 \subseteq C$ recovered(=removed) nominal items

$K_2 \subseteq J$ recovered(=removed) peak items



Definition 13 (A quadruple (C, J, K_1, K_2) is a Γ*-rrKP cover* if)

- $|J| \leq \Gamma$, $C \cap J = \emptyset$, and $|K_1| + |K_2| = k$ and
- $w^0(C \cup J) \geq c^0 + 1$ (first stage cover)
- or (C, K_1, J, K_2) is a second stage cover:

$$\sum_{i \in C \setminus K_1} \bar{w}_i + \sum_{i \in J \setminus K_2} (\bar{w}_i + \hat{w}_i) \geq c + 1$$

Theorem 14

Given a Γ*-rrKP cover* (C, K_1, J, K_2)*, the* Γ-rrKP cover inequality

$$\sum_{i \in C \cup J} x_i \leq |C \cup J| - 1$$

Theorem 15

Let $x \in \{0,1\}^n$. Then $x \in \mathcal{K}_D$ ($x \in \mathcal{K}_\Gamma$) if and only if x satisfies all minimal (Γ-)rrKP cover inequalities.

I.e., the minimal cover inequalities provide a formulation of the problem.

But, they do not provide a complete description of the convex hull of binary solutions.

Extended Cover Inequalities

Non-robust knapsack: Let $E(C) := \left\{ j \in N \ : \ w_j^0 \geq \max_{i \in C} w_i^0 \right\} \cup C$. Then the *Extended Cover inequality* for non-robust knapsack reads:

$$\sum_{i \in E(C)} x_i \leq |C| - 1$$

rrKP with discrete scenarios: A cover C w.r.t. scenario S can be extended with items whose weights exceed

(canonical extension) the weight of the $k+1$ highest-weight-item in C

(advanced extension)

1. the residual capacity according to the weights of the first $|C| - k - 1$ lowest-weight-items
2. the weight of the $k+2$ highest-weight-item in C

Theorem 16

Let S be a cover w.r.t scenario S and $E^S(C)$ its extension. Then

Extended Cover Inequalities

rrKP with Γ scenarios:

$$E(C, K_1, J, K_2) := \left\{ j \in N \ : \ \bar{w}_j \geq \max_{i \in C \setminus K_1} \bar{w}_i \text{ and } \bar{w}_j + \hat{w}_j \geq \max_{i \in J \setminus K_2} \bar{w}_i + \hat{w}_i \right\}$$
$$\cup \{ C \cup J \}$$

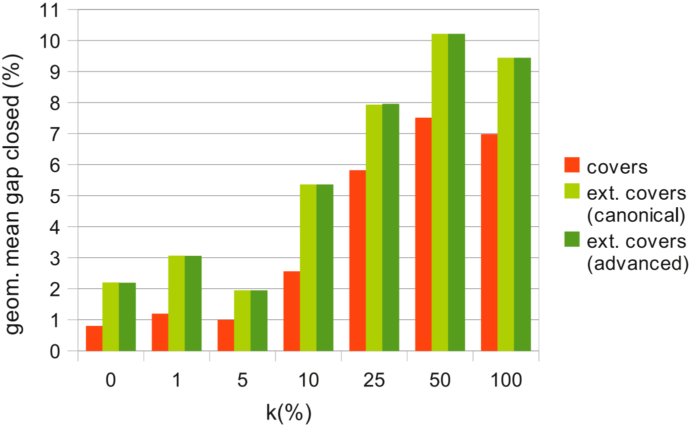
Theorem 17

Let (C, K_1, J, K_2) be a Γ cover and $E(C, K_1, J, K_2)$ its extension. Then

$$\sum_{i \in E(C, K_1, J, K_2)} x_i \leq |C \cup J| - 1$$

is valid for $\mathcal{K}_\Gamma$.

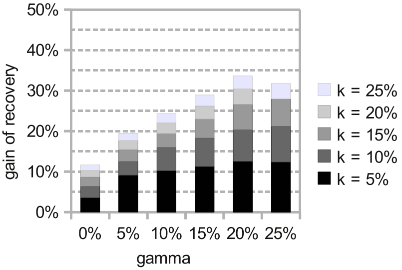
Gap closed by cover inequalities:

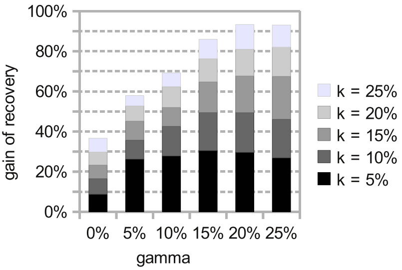


Note: the complete ILP formulation is used

- exact separation by ILP

Gain of Recovery:





geometric mean

observed maximum

For each instance, Γ,and k, the gain of recovery is determined by the objective value normalized to the corresponding case with $k = 0\,\%$.

Further Robustness Concepts

- Light Robustness: bound price of robustness of budget uncertainty and minimize weighted sum of constraint violations
- Distributionally Robust Optimization: given historical data, find solutions that are robust whatever probability distribution the data follows. *Marriage between stochastic and robust optimization*
- Minimax Regret Optimization or Robust deviation: Minimize largest possible difference between observed objective value of robust solution and optimal solution value (knowing uncertain parameters in advance)
- Relative robust deviation: Minimize largest possible ratio of robust deviation to the optimal objective value
- Multi-Band Robustness Multiple nested intervals with multiple decreasing Γ values

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