Robust Optimization & Network Design
Lecture 5

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Lehrstuhl II für Mathematik
RWTH AACHEN UNIVERSITY

1. Chance-Constraints and Γ-Robustness
2. Combinatorial Optimization with Uncertain Objective

This lecture:

- A connection between chance-constrained optimization and Γ-robustness.
- Solving robust combinatorial problems by a sequence of deterministic problems

Γ-Robust Counterpart

Robust Counterpart For $\Gamma \in \mathbb{Z}_+$:

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + \max_{S \subseteq \{1,\dots,n\}: |S| \leq \Gamma} \left(\sum_{j \in S} \hat{a}_{ij} x_j \right) \leq b_i \tag{1}$$

Let $J \subseteq \{1, \dots, n\}$ be the set of uncertain coefficients of the constraint $a^T x \leq b$.
Define

$$\beta(x^\star, \Gamma) := \max_{S \subseteq J, |S| \leq \Gamma} \left\{ \sum_{j \in S} \hat{a}_j x_j^\star \right\}$$

as maximum added value to the left hand side of $\bar{a}_j x^\star \leq b$.

Connection with Chance-constraints

Let $\zeta_j = \frac{a_j - \bar{a}_j}{\hat{a}_j}$ be random variables with values in $[-1, 1]$.

Theorem 1 (Bertsimas and Sim, 2004)

Let $x^\star \geq 0$ be an optimal solution of an ULO containing the robust counterpart

$$\sum_{j=1}^{n} \bar{a}_j x_j + \beta(x, \Gamma) \leq b$$

Further, let $S^\star$ be the index set defining $\beta(x^\star, \Gamma)$. Then,

$$\mathcal{P}\left(\sum_{j=1}^{n} a_j x_j^\star > b\right) \leq \mathcal{P}\left(\sum_{j \in J} \gamma_j \zeta_j \geq \Gamma\right)$$

with $\gamma_j = \begin{cases} 1 & \text{if } j \in S^\star \\ \frac{\hat{a}_j x_j^\star}{\hat{a}_r x_r^\star} & \text{if } j \in J \setminus S^\star \end{cases}$ *and* $r = \arg\min_{j \in S^\star} \hat{a}_j x_j^\star$.

Moreover, $\gamma_j \leq 1$ for all $j \in J \setminus S^\star$.

Theorem 2 (Bertsimas and Sim, 2004)

Let ζ_j, $j \in J$ be independent and symmetriccally distributed random variables in $[-1, 1]$. Then,

$$\mathcal{P}\left(\sum_{j\in J} \gamma_j \zeta_j \geq \Gamma\right) \leq \exp\left(-\frac{\Gamma^2}{2|J|}\right)$$

Example: $|J| = 100$, $\Gamma = 10 \Rightarrow \mathcal{P}(\text{violation}) \leq \exp(-\frac{1}{2}) \approx 0.6$
$|J| = 100$, $\Gamma = 20 \Rightarrow \mathcal{P}(\text{violation}) \leq \exp(-2) \approx 0.13$

Markov's Inequality

For a random variable X with finite expectation, it holds that

$$\mathcal{P}(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}$$

for all $a > 0$.

A better bound:

Theorem 3 (Bertsimas and Sim, 2004)

Let ζ_j, $j \in J$ be independent and symmetriccally distributed random variables in $[-1,1]$. Then,

$$\mathcal{P}\left(\sum_{j\in J}\gamma_j\zeta_j \geq \Gamma\right) \leq B(k,\Gamma) \qquad (2)$$

with $k = |J|$ and

$$B(k,\Gamma) = \frac{1}{2^k}\left\{(1-\mu)\binom{k}{\lfloor\nu\rfloor} + \sum_{\ell=\lfloor\nu\rfloor+1}^{k}\binom{k}{\ell}\right\}$$

where $\nu = \frac{1}{2}(\Gamma + k)$ and $\mu = \nu - \lfloor\nu\rfloor$.
Moreover, the bound (2) is tight whenever ζ_j has a discrete probability distribution with $\mathcal{P}(\zeta_j = 1) = \frac{1}{2}$ and $\mathcal{P}(\zeta_j = -1) = \frac{1}{2}$, $\Gamma \geq 1$ and $\Gamma + k$ even.
For $\Gamma = \theta\sqrt{k}$: $\lim_{k\to\infty} B(k,\Gamma) = 1 - \Phi(\theta)$ with $\Phi(\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{2}\right)dy$, the cumulative distribution function of the standard normal distribution.

Corollary 4 (Bertsimas & Sim, 2004)

Let $x^\star$ be an optimal solution of the Γ-robust counterpart. If a_j, $j = 1, \ldots, n$, are independent and symmetric distributed random variables in $[\bar{a}_j - \hat{a}_j, \bar{a}_j + \hat{a}_j]$, then

$$\mathcal{P}\left(\sum_{j=1}^{n} a_j x_i^\star > b\right) \leq B(n, \Gamma)$$

with

$$\lim_{n \to \infty} B(n, \Gamma) = 1 - \Phi\left(\frac{\Gamma}{\sqrt{n}}\right)$$

where $\Phi(.)$ is the CDF of the standard normal distribution.

Instead of the limit: $B(n, \Gamma) \approx 1 - \Phi\left(\frac{\Gamma - 1}{\sqrt{n}}\right)$

Choice of Γ

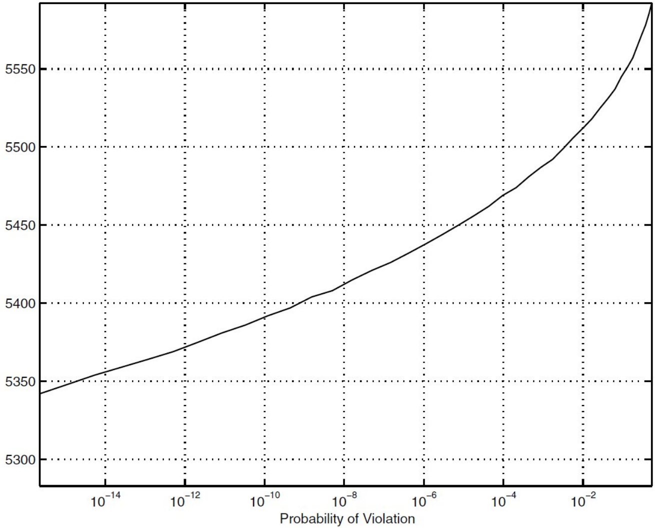
Choice of Γ as a function of n so that the probability of constraint violation is less than p%:

	Γ		
n	$p = 1$	$p = 0.5$	$p = 0.1$
5	5.0	5.0	5.0
10	8.4	9.1	10.0
100	24.3	26.8	31.9
200	33.9	37.4	44.7
1,000	74.6	82.5	98.7
2,000	105.0	116.2	139.2

Note: Result is independent of actual distribution of random variables a_{ij}, only symmetry and independence are required.

Violation vs. Price

Optimal value of the robust knapsack formulation as a function of the probability bound of constraint violation given in Equation (18).



Definition 5

Let $X \subseteq \{0,1\}^n$ for some $n \in \mathbb{Z}_+$. A combinatorial optimization problem is a problem of the form

$$\min \left\{ c^T x : x \in X \right\}$$

Assumption (for the moment):

X does not contain any uncertainty. Only c is uncertain!

Theorem 6 (Bertsimas and Sim, 2003)

Let $c_i \in [\bar{c}_i - \hat{c}_i, \bar{c}_i + \hat{c}_i]$ ($i \in N = \{1, \dots, n\}$) and uncertainty budget $\Gamma \in \mathbb{Z}_+$ define an Γ-uncertain combinatorial optimization (UCO) problem. Then, the robust counterpart of $\max\{c^T x : x \in X\}$ can be solved by solving $n+1$ deterministic problems of the form $\max\{d^T x : x \in X\}$.

Robust Counterpart of Γ-UCO

Robust Counterpart:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \bar{c}_i x_i + \Gamma\pi + \sum_{i=1}^{n} \rho_i \\ s.t. \quad & \pi + \rho_i \geq \hat{c}_i x_i && \forall i \in N \\ & \pi, \rho_i \geq 0 && \forall i \in N \\ & x \in X \end{aligned}$$

Question: Given $x \in X$, what is the minimum contribution of $\Gamma\pi + \sum_{i=1}^{n} \rho_i$?

Answer:

1. Find a permutation σ of the items such that $\hat{c}_{\sigma(1)} x_{\sigma(1)} \geq \hat{c}_{\sigma(2)} x_{\sigma(2)} \geq \ldots \geq \hat{c}_{\sigma(n)} x_{\sigma(n)}$
2. Set $\pi := \hat{c}_{\sigma(\Gamma)} x_{\sigma(\Gamma)}$
3. Set $\rho_i := \max\left(0, \hat{c}_i x_i - \pi\right)$
   I.e., $\rho_i = 0$ for all $i : \sigma^{-1}(i) \geq \Gamma$, and $\hat{c}_i x_i - \pi$ if $\sigma^{-1}(i) < \Gamma$

Robust Counterpart of Γ-UCO

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2. Set $\pi := \hat{c}_{\sigma(\Gamma)}x_{\sigma(\Gamma)}$
3. Set $\rho_i := \max(0, \hat{c}_i x_i - \pi)$

Corollary 7

Given $x \in X \subseteq \{0,1\}^n$, the optimal value of π is one of the values $\{0\} \cup \{\hat{c}_1, \ldots, \hat{c}_n\}$.

Corollary 8

Given $x \in X \subseteq \{0,1\}^n$, the optimal solution value is determined by

$$\sum_{i=1}^{n} \bar{c}_i x_i + \max\left\{\sum_{i=1}^{n} \hat{c}_i x_i, \max_{k=1,\ldots,n}\left\{\Gamma \hat{c}_k + \sum_{i=1}^{n}(\hat{c}_i - \hat{c}_k)^+ x_i\right\}\right\}$$

where $(a)^+ := \max\{0, a\}$.

Robust Counterpart of Γ-UCO

Theorem 9 (Bertsimas and Sim, 2003)

The UCO $\min\{c^T x : x \in X\}$ *can be solved by solving for all* $\pi \in \{0, \hat{c}_1, \ldots, \hat{c}_n\}$ *the following CO problem*

$$\Gamma\pi + \min\{\sum_{i=1}^{n}(\bar{c}_i + (\hat{c}_i - \pi)^+)x_i : x \in X\}$$

and selecting the cheapest solution.

Corollary 10

- *If a CO problem can be solved in polynomial time (e.g., shortest path, min spanning tree, min cost flow, max matching) the UCO (with uncertain objective) can be solved in polynomial time*
- *The knapsack problem with uncertain objective can be solved in* $O(n^2 B)$.

Exercise 1

Consider the Γ-robust knapsack problem

$$\max \left\{ \sum_{i=1}^{n} c_i x_i : \sum_{i=1}^{n} a_i x_i \leq b, x_i \in \{0,1\} \right\}$$

where c_i are random variables, $c_i \in [\bar{c}_i - \hat{c}_i, \bar{c}_i + \hat{c}_i]$.

Let $n = 5$, $b = 250$, $\bar{c} = \begin{pmatrix} 50 \\ 30 \\ 45 \\ 25 \\ 70 \end{pmatrix}$, $\hat{c} = \begin{pmatrix} 18 \\ 8 \\ 15 \\ 4 \\ 33 \end{pmatrix}$, and $a = \begin{pmatrix} 61 \\ 67 \\ 64 \\ 52 \\ 113 \end{pmatrix}$.

Determine the optimal solution for $\Gamma = 0, 1, 2, 3, 4,$ and 5.

Corollary 11

The knapsack problem with uncertain objective can be solved in $O(n^2 B)$.

Theorem 12

The knapsack problem with uncertain weight can be solved in $O(n^2 B)$.

Theorem 13 (Monaci et al. (2013))

The knapsack problem with uncertain weight can be solved in $O(n \Gamma B)$.

Exercise 2

The knapsack problem $\max\{c^T x : a^T x \leq B, x \in \{0,1\}^n\}$ can be solved by a dynamic programming algorithm in $O(nB)$ time. For this, the function

$$f(k,d) := \max\left\{\sum_{i=1}^{k} c_i x_i : \sum_{i=1}^{k} a_i x_i = d, x_i \in \{0,1\}\right\}$$

is solved for all $k \in \{0,\ldots,n\}$ and $d \in \{0,\ldots,b\}$.
Develop a dynamic programming algorithm for the Γ-robust knapsack problem with uncertain weights. What is the running time?

Theorem 14 (Bertsimas and Sim, 2003)

If a CO problem can be approximated in polynomial time with approximation factor α, the UCO (with uncertain objective) can be approximated in polynomial time with approximation factor α.

Remark: The approximation should hold for all possible inputs. In case of the symm. TSP under triangle inequality, $\alpha = \frac{3}{2}$, but it has to be guaranteed that also with (some of) the deviations, the triangle inequality still holds.

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