Discrete Optimization under Uncertainty
Lecture 3

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Observation

In the knapsack example, normal distribution of the weights was assumed. What if, the weights are distributed differently, or unknown?

Uncertain Linear Program

An Uncertain Linear Optimization problem (ULO) is a collection of linear optimization problems (instances)

$$\left\{\min\{c^T x + d : Ax \leq b\}\right\}_{(c,d,A,b)\in\mathcal{U}}$$

where all input data stems from an uncertainty set $\mathcal{U} \subset \mathbb{R}^{m+1\times n+1}$.

Perturbation Set

The uncertainty set $\mathcal{U}$ is usually described by an affine parameterization: a perturbation vector ζ from a perturbation set $\mathcal{Z}$ describes all possible deviations from a nominal matrix $D_0 = \begin{pmatrix} c_0^T & d_0 \\ A_0 & b_0 \end{pmatrix}$:

$$\mathcal{U} = \left\{ D \in \mathbb{R}^{m+1 \times n+1} : D = D_0 + \sum_{\ell=1}^{L} \zeta_\ell D_\ell : \zeta \in \mathcal{Z} \subset \mathbb{R}^L \right\}$$

The perturbation set $\mathcal{Z}$ describes how the deviations can be combined.

Example: production planning

Products: DrugI, DrugII containing an active agent A

Parameter	DrugI	DrugII
Selling price, $ per 1000 packs	6,200	6,900
Content of agent A, g per 1000 packs	0.5	0.6
Manpower required, hours per 1000 packs	90	100
Equipment required, hours per 1000 packs	40	50
Operational costs, $ per 1000 packs	700	800

Contents of Raw material:

Raw material	Purchasing price, $ per kg	Content of Agent A, g per kg
RawI	100.00	0.01 $\pm$ 0.5%
RawII	199.90	0.02 $\pm$ 2%

Resources:

Budget, $	Manpower, hrs	Equipment, hrs	Capacity of raw materials storage, kg
100,000	2,000	800	1,000

Example: production planning

Decision vector: $x = [RawI; RawII; DrugI; DrugII]$
Nominal data:

$$D_0 = \begin{pmatrix} 100 & 199.9 & -5500 & -6100 & 0 \\ -0.01 & -0.02 & 0.5 & 0.6 & 0 \\ 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 90 & 100 & 2000 \\ 0 & 0 & 40 & 50 & 800 \\ 100.0 & 199.9 & 700 & 800 & 100000 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Perturbation matrices:

$$D_1 = 5.0 \cdot 10^{-5} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; D_2 = 4.0 \cdot 10^{-4} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Perturbation set:

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^2 : -1 \leq \zeta_1, \zeta_2 \leq 1\}$$

Typical Perturbation Sets

Typical perturbation sets are:

- the unit box (interval uncertainty)

$$\left\{\zeta \in \mathbb{R}^L : -1 \leq \zeta_\ell \leq 1 \quad \forall \ell = 1, \dots, L\right\}$$

- the discrete scenarios

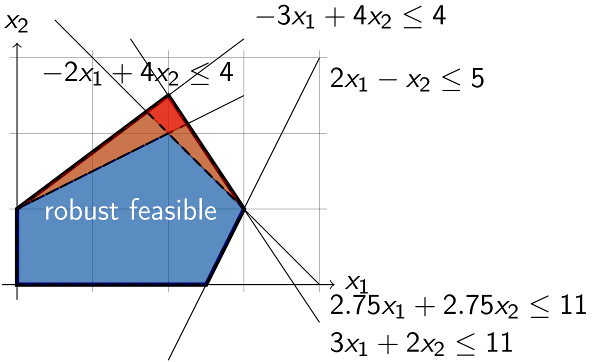
$$\left\{\zeta \in \mathbb{R}^L : \sum_{\ell=1}^{L} \zeta_\ell \leq 1, 0 \leq \zeta_\ell \leq 1 \quad \forall \ell = 1, \dots, L\right\}$$

- the Eucledian ball with unit radius

$$\left\{\zeta \in \mathbb{R}^L : ||\zeta||^2 = \zeta^T \zeta \leq 1\right\}$$

In One-Stage Robust Optimization, we only consider ULOs with the following characteristics:

1. All decision variables represent here and now decisions; they should be assigned specific numerical values as a result of solving the problem before the actual data "reveals itself."
2. The decision maker is fully responisble for consequences of the decisions to be made when, and only when, the actual data is within the prespecified uncertainty set $\mathcal{U}$.
3. The constraints $Ax \leq b$ are hard – we cannot tolerate violations of constraints, even small ones, when the data is in $\mathcal{U}$.



$ax_1 + 4x_2 \leq 4$ with $a \in [-3, -2]$

$bx_1 + cx_2 \leq 11$ with $b \in [2.75, 3]$ and $c \in [2, 2.75]$

Select optimal solution among *robust* solutions!

ULO $\left\{\min\{c^Tx + d : Ax \leq b\}\right\}_{(c,d,A,b)\in\mathcal{U}}$

Robust feasible solution

A vector $x \in \mathbb{R}^n$ is robust feasible for ULO if

$$Ax \leq b \qquad \forall (c, d, A, b) \in \mathcal{U}$$

Robust solution value

Given a vector $x \in \mathbb{R}^n$, the robust solution value $\hat{c}(x)$ is defined as

$$\hat{c}(x) := \sup_{(c,d,A,b)\in\mathcal{U}} \left(c^Tx + d\right)$$

Robust Counterpart

The robust counterpart of an ULO is the optimization problem

$$\min\left\{\hat{c}(x) : x \text{ is robust feasible}\right\}$$

Example

Let $\left\{\min\{c^T x : Ax \leq b, x \geq 0\}\right\}_{(c,A,b)\in\mathcal{U}}$ be an ULO with uncertain right-hand-side

$$b \in [\bar{b}, \bar{b} + \hat{b}]$$

uncertain matrix A,

$$a_{ij} \in [\bar{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$$

but certain objective vector c.

The robust counterpart can be written as

$$\min\{c^T x : (\bar{A} + \hat{A})x \leq \bar{b}, x \geq 0\}$$

Let A be a $m \times n$ matrix. Consider the following uncertain linear optimization problem:

$$\min_x \{c^T x : Ax \leq b\},$$

under the uncertainty:

$$\mathcal{U} = \{(c, A, b) : |c_j - \bar{c}_j| \leq \sigma_j, \left|A_{ij} - \bar{A}_{ij}\right| \leq \alpha_{ij}, \left|b_i - \bar{b}_i\right| \leq \beta_i, \forall i, j\},$$

where $\bar{c}_j$, etc. denotes the nominal data.

Reduce the robust counterpart of the problem to a linear program with

- m constraints (not counting the non-negativity constraints) and
- $2n$ nonnegative variables.

Answer: x is free, and has to be replaced by $x = x^+ - x^-$ with $x^+ \geq 0$, $x^- \geq 0$

Observation

If the objective is certain, the robust counterpart can be constructed row-wise, i.e.,

- keep the objective
- replace every constraint $a_i^T x \leq b_i$ by its robust counterpart

$$a_i^T x \leq b_i \qquad \forall (a_i, b_i) \in \mathcal{U}_i$$

where

$$\mathcal{U}_i := \left\{ (\tilde{a}_i, \tilde{b}_i) \in \mathbb{R}^{n+1} : \exists (A, b) \in \mathcal{U} \text{ with } A_{i\cdot} = \tilde{a}_i, b_i = \tilde{b}_i \right\}$$

Note: the robust counterpart does not change if $\hat{\mathcal{U}} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_m$ instead of $\mathcal{U}$ is used.

Wlog: Objective vector c is certain!

Corollary

If only the right hand side b is uncertain, the robust counter part reads

$$Ax \leq \bar{b}$$

with $\bar{b}_i = \min\{b_i : (A, b, c) \in \mathcal{U}\}$.

Max-Flow with uncertain capacities:

- Take minimum capacity on every arc, and solve the max flow problem.

Min-Cut with uncertain capacities:

- Objective vector c is uncertain! Requires solving of a new problem.

Corollary: Robust Max-Flow $\neq$ Robust Min-Cut

General framework

By the earlier observation, we can focus on a single uncertainty-affected linear inequality

$$\{a^T x \leq b\}_{[a;b] \in \mathcal{U}} \tag{1}$$

with uncertainty set

$$\mathcal{U} = \left\{ [a;b] = [a^0; b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell; b^\ell] : \zeta \in \mathcal{Z} \right\} \tag{2}$$

The robust counterpart reads

$$a^T x \leq b \quad \forall \left([a;b] = [a^0; b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell; b^\ell] : \zeta \in \mathcal{Z} \right) \tag{3}$$

Let

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : ||\zeta||_\infty \leq 1\}$$

thus a box around the origin, also called interval uncertainty.
In this case, (3) reads

$$[a^0]^T x + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell]^T x \leq b^0 + \sum_{\ell=1}^{L} \zeta_\ell b^\ell \qquad \forall \zeta : ||\zeta||_\infty \leq 1$$

$$\Leftrightarrow \sum_{\ell=1}^{L} \zeta_\ell \left[[a^\ell]^T x - b^\ell \right] \leq b^0 - [a^0]^T x \qquad \forall \zeta : |\zeta_\ell| \leq 1, \ell = 1, \ldots, L$$

$$\Leftrightarrow \sum_{\ell=1}^{L} \max_{-1 \leq \zeta_\ell \leq 1} \left[\zeta_\ell \left[[a^\ell]^T x - b^\ell \right] \right] \leq b^0 - [a^0]^T x$$

$$\Leftrightarrow \sum_{\ell=1}^{L} |[a^\ell]^T x - b^\ell| \leq b^0 - [a^0]^T x$$

Now,

$$\sum_{\ell=1}^{L} |[a^\ell]^T x - b^\ell| \leq b^0 - [a^0]^T x$$

can be easily reformulated by a system of linear inequalities:

$$[a^0]^T x + \sum_{\ell=1}^{L} u_\ell \leq b^0$$
$$-u_\ell \leq [a^\ell]^T x - b^\ell \leq u_\ell \quad \forall \ell = 1, \ldots, L$$

Knapsack with Interval Uncertainty

Knapsack with n Items, profits c_i, uncertain weights $a_i \in [\underline{a}_i, \bar{a}_i]$, and capacity b

Exercise:

1. Define $[a^\ell; b^\ell]$ for all $\ell = 1, \ldots, L$ (how large is L?)
2. Simplify $\max_{-1 \leq \zeta_\ell \leq 1} \zeta_\ell([a^\ell]^T x - b^\ell)$
3. How does the Robust Counterpart look like?

Let

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : ||\zeta||_2 \leq \Omega\}$$

thus a ball of radius Ω around the origin.

In this case, (3) reads

$$[a^0]^T x + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell]^T x \leq b^0 + \sum_{\ell=1}^{L} \zeta_\ell b^\ell \qquad \forall \zeta : ||\zeta||_2 \leq \Omega$$

$$\Leftrightarrow \max_{||\zeta||_2 \leq \Omega} \left[\sum_{\ell=1}^{L} \zeta_\ell \left[[a^\ell]^T x - b^\ell \right] \right] \leq b^0 - [a^0]^T x$$

$$\Leftrightarrow \Omega \sqrt{\sum_{\ell=1}^{L} ([a^\ell]^T x - b^\ell)^2} \leq b^0 - [a^0]^T x$$

Polyhedral Uncertainty Set

Let

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : P\zeta \leq q\}$$

with $P \in \mathbb{R}^{M \times L}$, $q \in \mathbb{R}^M$, i.e., $\mathcal{Z}$ is described by a polyhedron.
In this case, (3) reads

$$[a^0]^T x + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell]^T x \leq b^0 + \sum_{\ell=1}^{L} \zeta_\ell b^\ell \qquad \forall \zeta : P\zeta \leq q$$

$$\Leftrightarrow \max_{\zeta : P\zeta \leq q} \left[\sum_{\ell=1}^{L} \zeta_\ell \left[[a^\ell]^T x - b^\ell \right] \right] \leq b^0 - [a^0]^T x \tag{4}$$

Given x (fixed), feasability of (4) can be checked by solving the LP:

$$z(x) = \max \sum_{\ell=1}^{L} \left[[a^\ell]^T x - b^\ell \right] \zeta_\ell$$

$$s.t. P\zeta \leq q$$

If $z(x) \leq b^0 - [a^0]^T x$, then x is robust feasible, otherwise not.

Discrete Uncertainty Set

Let $[a; b]$ be taken from a discrete set of N possible realizations $\{[a^i; b^i]\}_{i=1,\dots,N}$.

Approach 1:

- Define $[a^1; b^1]$ as the nominal case
- Set $[\tilde{a}^i; \tilde{b}^i] := [a^i - a^1; b^i - b^1]$ for all $i = 2, \dots, N$
- Define the perturbation set

$$\mathcal{Z} = \left\{ \zeta \in \mathbb{R}^{N-1} : \zeta_i \in \{0, 1\} [0, 1], \sum_{i=2}^{N} \zeta_i \leq 1 \right\}$$

- Construct Robust Counterpart

Discrete Uncertainty Set

Let $[a; b]$ be taken from a discrete set of N possible realizations $\{[a^i; b^i]\}_{i=1,\ldots,N}$.

Approach 2:

- Compute a polyhedral description of the convex hull of $\{[a^i; b^i], i = 1, \ldots, N\}$
- I.e., $\{\zeta \in \mathbb{R}^{n+1} : P\zeta \leq q\}$ has as extreme points $[a^i; b^i]$
- Define the perturbation set

$$\mathcal{Z} = \left\{\zeta \in \mathbb{R}^{n+1} : P\zeta \leq q\right\}$$

  and the uncertainty set

$$\mathcal{U} = \left\{[a; b] = [0; 0] + \sum_{i=0}^{n} \zeta_i [e^i; 0] + \zeta_{n+1}[\underline{0}, 1]\right\}$$

- Construct Robust Counterpart

Discrete Uncertainty Set

Let $[a; b]$ be taken from a discrete set of N possible realizations $\{[a^i; b^i]\}_{i=1,\dots,N}$.

Approach 3:

- Replace $a^T x \leq b$ by

$$[a^i]^T x \leq b^i \quad \forall i = 1, \dots, N$$

Definition of $\mathcal{Z}$ and $\mathcal{U}$ is sometimes unnecessarily difficult!

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