Algorithmic Graph Theory: How hard is your combinatorial optimization problem?

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Lecture 11

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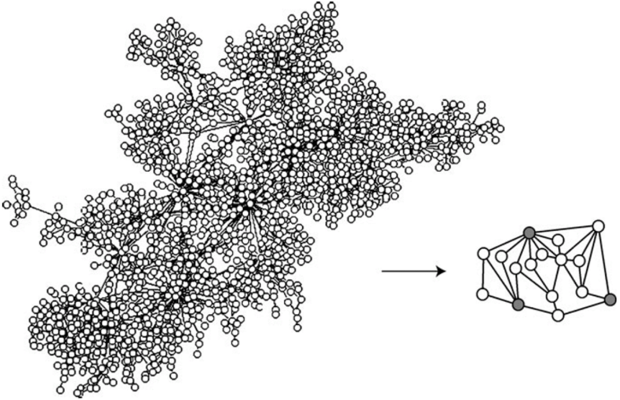
Lehrstuhl II für Mathematik

Yesterday: Dominating Set



Power of kernelization

Algorithm mis1(G).
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Output: The maximum cardinality of an independent set of G.

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Fig. 1.2 Algorithm mis1 for MAXIMUM INDEPENDENT SET

What is the running time of MIS1 ?

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In the context of exact exponential algorithms, we write

$$f(n) = O^*(g(n))$$

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We write $O(\sqrt{2})^n \cdot poly(n)) = O^*(\sqrt{2}^n) = O(1.4143^n)$.
Note: Not only running time plays a role, also the memory requirements.

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Correctness of a branching is often easy to prove; the analysis more difficult.

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$$T(n) \leq T(n - t_1) + T(n - t_2) + \ldots + T(n - t_r)$$

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In worst-case, equality holds, and we search c such that $T(n) = c^n$.

Example: Branching vector $b = (1, 1)$:

Branching factor

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Theorem

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$$x^n - x^{n-t_1} - x^{n-t_2} - \ldots - x^{n-t_r} = 0.$$

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- *If* $t_1 > t_1'$, *then* $\tau(t_1, \ldots, t_r) < \tau(t_1', t_2, \ldots, t_r)$

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$\tau(3,3) < 1.2600$, $\tau(2,4) = \tau(4,2) < 1.2712$, $\tau(5,1) = \tau(1,5) < 1.3248$

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Lemma

If vertices of degree 0 and 1 are preprocessed, then MIS1 *has running time* $O^*(\sqrt[3]{3}) = O(1.4423^n)$.

Multiple Branching Rules

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Let (i, j) *be a branching vector and* (k, l) *the branching vector for the subproblem of siz* $n - i$. *Then,* $(i + k, i + l, j)$ *is a branching vector for the combined branching.*

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Let G *be a non-connected graph with* $C \subset V$ *defining a connected component of* G*. Then,* $\alpha(G) = \alpha(G - C) + \alpha(G[C])$*.*

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Lemma (Mirror Branching)

Let $G = (V, E)$ and $v \in V$. Then

$$\alpha(G) = \max\{1 + \alpha(G - N[v]), \alpha(G - (\{v\} \cup M(v))\}.$$

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Lemma (Separator Branching)

Let $G = (V, E)$ and S separator of G. Let $\mathcal{I}(S)$ be the collection of all independent sets of S. Then,

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See page 27, Fomin & Kratsch, 2010.

Theorem

MIS2 *solves the max. independent set problem in* $O(1.2786^n)$.

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Algorithm mis3**(G).**
Input: A graph $G = (V, E)$.
Output: The maximum cardinality of an independent set of G.

1 **if** $\exists v \in V$ with $d(v) = 0$ **then**
  return $1 + \texttt{mis3}(G \setminus v)$

2 **if** $\exists v \in V$ with $d(v) = 1$ **then**
  return $1 + \texttt{mis3}(G \setminus N[v])$

3 **if** $\Delta(G) \geq 3$ **then**
  choose a vertex v of maximum degree in G
  return $\max(1 + \texttt{mis3}(G \setminus N[v]), \texttt{mis3}(G \setminus v))$

4 **if** $\Delta(G) \leq 2$ **then**
  compute $\alpha(G)$ using a polynomial time algorithm
  return $\alpha(G)$

Fig. 6.1 Algorithm mis3 for MIS

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- Vertices of degree ≤ 1 have weight 0
- Vertices of degree 2 have weight $w_2 \in [0,1]$
- Vertices of degree ≥ 3 have weight 3

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With measure $K_2(G') := w_2 n_2 + n_{\geq 3}$, MIS3 has running time $O(1.3248^n)$.

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Let $k(G') = \sum_{i=0}^{n} w_i n_i$ with $w_i \in [0,1]$ and $n_i = |\{v \in V : deg(v) = i\}|$. If $w_0 = w_1 = 0$, $w_2 = 0.596601$, $w_3 = 0.928643$ and $w_t = 1$ for $t \geq 4$, then MIS3 has running time $O(1.2905^n)$.

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Note: There exists an algorithm with running time $O(1.2209^n)$.

Example (MIN DOMINATING SET - MDS)

Given: Graph $G = (V, E)$

Find: Dominating Set $D \subseteq V$ of minimum cardinality, i.e., $\forall v \in V$: $N[v] \cap D \neq \emptyset$.

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Given: Ground set U, collection $\mathcal{S}$ of nonempty subsets of U.

Find: Set Cover of $(U, \mathcal{S})$, i.e., a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $\cup_{s \in \mathcal{S}'} s = U$. Minimize $|\mathcal{S}'|$.

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For a MSC instance it holds:

- *If for two distinct sets S and R, $S \subseteq R$, then there exists a MSC not containing S.*

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Lemma

A MSC instance with $|S| = 2$ for all $S \in \mathcal{S}$ can be solved in polynomial time.

Solving MSC

Define $del(S, \mathcal{S}) = \{T : T = R \setminus S \neq \emptyset, R \in \mathcal{S}\}$

Algorithm msc($\mathcal{S}$).
Input: A collection $\mathcal{S}$ of subsets of a universe $\mathcal{U}$.
Output: The minimum cardinality of a set cover of $\mathcal{S}$.

```
1 if |𝒮| = 0 then
    return 0
2 if ∃S,R ∈ 𝒮 with S ⊆ R then
    return msc(𝒮\{S})
3 if ∃u ∈ 𝒰(𝒮) such that there is a unique S ∈ 𝒮 with u ∈ S then
    return 1 + msc(del(S,𝒮))
4 choose a set S ∈ 𝒮 of maximum cardinality
5 if |S| = 2 then
    return poly-msc(𝒮)
6 if |S| ≥ 3 then
    return min(msc(𝒮\{S}), 1 + msc(del(S,𝒮)))
```

Fig. 6.4 Algorithm msc for MSC

Solving MSC

Define $del(S, \mathcal{S}) = \{T : T = R \setminus S \neq \emptyset, R \in \mathcal{S}\}$

Algorithm msc($\mathcal{S}$).
Input: A collection $\mathcal{S}$ of subsets of a universe $\mathcal{U}$.
Output: The minimum cardinality of a set cover of $\mathcal{S}$.

1 **if** $|\mathcal{S}| = 0$ **then**
  return 0
2 **if** $\exists S, R \in \mathcal{S}$ with $S \subseteq R$ **then**
  return msc($\mathcal{S} \setminus \{S\}$)
3 **if** $\exists u \in \mathcal{U}(\mathcal{S})$ such that there is a unique $S \in \mathcal{S}$ with $u \in S$ **then**
  return $1 +$ `msc`(del($S, \mathcal{S}$))
4 choose a set $S \in \mathcal{S}$ of maximum cardinality
5 **if** $|S| = 2$ **then**
  return `poly-msc`($\mathcal{S}$)
6 **if** $|S| \geq 3$ **then**
  return min(`msc`($\mathcal{S} \setminus \{S\}$), $1 +$ `msc`(del($S, \mathcal{S}$)))

Fig. 6.4 Algorithm `msc` for `MSC`

Theorem

Algorithm MSC *solves the Minimum Set Cover problem in* $O(1.2353^{|\mathcal{S}|+|U|})$.

Back to MDS

Corollary

Minimum Dominating Set can be solved in $O(1.2353^{2n}) = O(1.5259^n)$.

Theorem

The worst-case running time of MSC for Min. Dominating Set is $\Omega(2^{\frac{n}{3}}) = \Omega(1.2599^n)$.

Dynamic Programming for TSP

Directed Feedback Arc Set

Example

Directed Feedback Arc Set Let $G = (V, A)$ be a directed graph. A feedback arc set is a subset of the arcs $F \subseteq A$ such that $(V, A \setminus F)$ is acyclic, i.e., every directed cycle in G contains at least one arc from F.

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Definition

A topological ordering of a directed graph $G = (V, A)$ is an ordering $\pi : V \to \{1, \ldots, n\}$ (with $n = |V|$) such that $\pi(u) < \pi(v)$, $\forall (u, v) \in A$.

All arcs are directed from left to right.

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All arcs are directed from left to right.

Lemma

Let $G = (V, A)$ be a directed graph and $w : A \to \mathbb{Z}_+$. Let $k \geq 0$, integer. There exists a feedback arc set F with weight $\sum_{a \in F} w_a \leq k$ if and only if there exists a linear ordering π of V s.t. $\sum_{(x,y) \in A: \pi(x) > \pi(y)} w(x, y) \leq k$.

π is a topological ordering of $(V, A \setminus F)$.

Theorem

The DIRECTED FEEDBACK ARC SET problem can be solved in $O(nm2^n) = O^*(2^n)$, *where* $n = |V|$ *and* $m = |A|$.

DFAS & Treewidth

Theorem

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Theorem

The treewidth of a graph with n *vertices can be determined in* $O^*(2^n)$ *time and* $O^*(2^n)$ *memory.*

Theorem

The treewidth of a graph with n *vertices can be determined in* $O^*(2.9512^n)$ *time and polynomial memory.*

Example (k-COLORABILITY)

Let $G = (V, E)$ be a graph and k integer. A k-coloring of G is an assignment $c : V \rightarrow \{1, \ldots, k\}$ such that $c(v) \neq c(w)$ for all $vw \in E$. The chromatic number $\chi(G)$ is the minimum k for which a k-coloring exists.

COLORABILITY

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k-COLORABILITY can be solved in $O^*(n^n) = O^*(2^{n \log n})$.

Theorem

$\chi(G)$ can be computed in $O^*((1 + \sqrt[3]{3})^n) = O(2.4423^n)$ with dynamic programming.

Algorithmic Graph Theory: How hard is your combinatorial optimization problem?

Arie M.C.A. Koster

Lecture 11

Clemson, June 14, 2017

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