Algorithmic Graph Theory: How hard is your combinatorial optimization problem?

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Lecture 9

Clemson, June 13, 2017

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1 Parameterized Problems
2 W-Hierarchy
3 Designing Parameterized Algorithms

Definition

A parameterized problem is a pair (Π, κ), where Π is a decision problem with set of instances $\mathcal{I}$ and $\kappa : \mathcal{I} \to \mathbb{N}$ a so-called parameter, a in polynomial time (in the size of $\mathcal{I}$) computable function.

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Parameterized problems are denoted by "p-" if parameterized by its "objective".

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Given: $G = (V, E)$ and integer $k \in \mathbb{N}$
Parameter: k
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Given: Graph $G = (V, E)$ and integer $k \in \mathbb{N}$
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Given: Graph $G = (V, E)$ of bounded treewidth and integer $k \in \mathbb{N}$
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Let (Π, κ) be a parameterized problem, $\mathcal{I}$ its set of instances.

- An algorithm A is called fixed parameter tractable (FPT) w.r.t. a parameter κ, if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ and a polynomial p, such that for every instance $I \in \mathcal{I}$, the running time of A is bounded by

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Theorem

p-TREEWIDTH $\in \mathcal{FPT}$

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p-VERTEX COVER $\in \mathcal{FPT}$

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Examples

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Note: Every problem $\Pi \in \mathcal{P}$ is with every parameterization κ in $\mathcal{FPT}$.
Note: Instead of multiplication, FPT can also be defined equivalently by

$$f(\kappa(I)) + p(|I|)$$

or the combination

$$g(\kappa(I)) + f(\kappa(I)) \cdot p(|I| + \kappa(I)).$$

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Definition

Let (Π, κ) be a parameterized problem and $k \in \mathbb{N}$. Then, the *k-th slice* of (Π, κ) is the classical decision problem Π restricted to the instances I having $\kappa(I) = k$.

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Example (p-PARTITION IN INDEPENDENT SETS)

Given: $G = (V, E)$, integer $k \in \mathbb{N}$.
Parameter: k
Question: Does V have a partition in k independent sets?

Theorem

Let (Π, κ) *be a parameterized problem and* $k \in \mathbb{N}$. *Is* (Π, κ) *fixed parameter tractable, then the* k*-th slice* (Π, κ) *can be solved in polynomial time.*

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Theorem

Let (Π, κ) be a parameterized problem and $k \in \mathbb{N}$. Is (Π, κ) fixed parameter tractable, then the k-th slice (Π, κ) can be solved in polynomial time.

Corollary

Unless $\mathcal{P} = \mathcal{NP}$, p-PARTITION IN INDEPENDENT SETS$\notin \mathcal{FPT}$.

Theorem

Let (Π, κ) be a parameterized problem and $k \in \mathbb{N}$. Is (Π, κ) fixed parameter tractable, then the k-th slice (Π, κ) can be solved in polynomial time.

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Unless $\mathcal{P} = \mathcal{NP}$, p-PARTITION IN INDEPENDENT SETS$\notin \mathcal{FPT}$.

Note: If all slices can be solved in polynomial time, it is not yet clear that the problem is in $\mathcal{FPT}$.

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Algorithm `mis1`**(G).**

Input: Graph $G = (V, E)$.

Output: The maximum cardinality of an independent set of G.

if $|V| = 0$ **then**
  return 0

choose a vertex v of minimum degree in G

return $1 + \max\{\texttt{mis1}(G \setminus N[y]) : y \in N[v]\}$

Fig. 1.2 Algorithm `mis1` for MAXIMUM INDEPENDENT SET

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Fig. 1.2 Algorithm `mis1` for Maximum Independent Set

If k is added, a running time of $O((\Delta(G) + 1)^k n) = O(p(n))$ can be achieved (for fixed k).

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Fig. 1.2 Algorithm mis1 for MAXIMUM INDEPENDENT SET

If k is added, a running time of $O((\Delta(G)+1)^k n) = O(p(n))$ can be achieved (for fixed k).
This is not an FPT-algorithm! ($f(k)$ depends on $\Delta(G)$)

Example (p-deg-INDEPENDENT SET)

Given: Graph $G = (V, E)$ and integer $k \in \mathbb{N}$
Parameter: $k + \Delta(G)$
Question: Does G have an independent set of size at least k?

Example (cont.)

Example (p-deg-INDEPENDENT SET)

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Corollary

p-deg-INDEPENDENT SET $\in \mathcal{FPT}$

For p-INDEPENDENT SET, the algorithm has running time $O(n^{k+1})$.

$\mathcal{XP}$-Algorithm

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Definition

Let (Π, κ) be a parameterized problem with $\mathcal{I}$ the set of instances.

- An algorithm A is called $\mathcal{XP}$-algorithm w.r.t. a parameterization $\kappa : \mathcal{I} \to \mathbb{N}$ if there exists computable functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for every instance $I \in \mathcal{I}$ the running time of A is bounded by

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Yes, we can!

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A parameterized problem (Π_1, κ_1) reduces parameterized to a parameterized problem (Π_2, κ_2) if there exists a function $r : \mathcal{I}_1 \to \mathcal{I}_2$ such that

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- p-INDEPENDENT SET reduces parameterized to p-CLIQUE

Theorem

Let (Π_1, κ_1) *and* (Π_2, κ_2) *be parameterized problems. If* (Π_1, κ_1) *reduces parameterized to* (Π_2, κ_2)*, and* $(\Pi_2, \kappa_2) \in \mathcal{FPT}$*, then* $(\Pi_1, \kappa_1) \in \mathcal{FPT}$*.*

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Given: a boolean formula and integer $k \in \mathbb{N}$
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Question: Is the boolean formula satisfiable with at least k variables set to TRUE?

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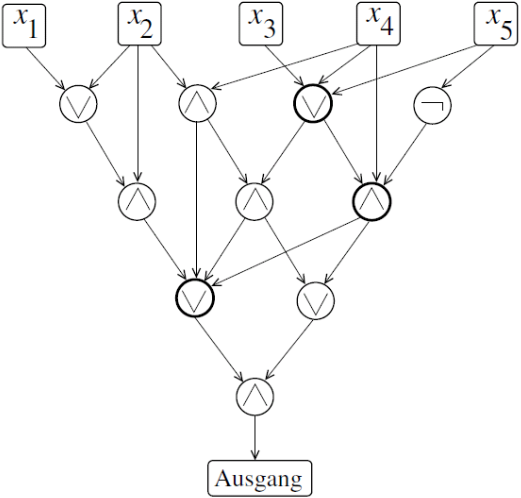
Definition (WEIGHTED WEFT-t-DEPTH-d SAT)

Given: A Boolean formula of depth at most d and weft at most t, and a number k. The depth is the maximal number of gates on any path from the root to a leaf, and the weft is the maximal number of gates of fan-in at least three on any path from the root to a leaf.
Question: Is the boolean formula satisfiable with k variables set to TRUE?



Boolean circuit with weft=3 and depth=5

Es is conjuctured that boolean formula with high weft are more difficult than those with low weft.

W-Hierarchy

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Definition

The W-Hierarchy consists of the complexity classes $W[t]$, $t \geq 1$. A parameterized problem (Π, κ) is a member of $W[t]$ if it can be reduced parameterized to p-WEIGHTED WEFT-t-DEPTH-d SAT for some $d \in \mathbb{N}$.

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Example (p-DOMINATING SET)

Given: Graph $G = (V, E)$, integer $k \in \mathbb{N}$
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Question: Does G have a dominating set of size at most k, i.e., a subset of the vertices $S \subseteq V$ such that for all vertices $v \in V$: $N[v] \cap S \neq \emptyset$.

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Lemma

p-DOMINATING SET $\in W[2]$

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Corollary: p-WEIGHTED WEFT-t-DEPTH-d SAT is $W[t]$-complete (by definition).

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Theorem

- *p-INDEPENDENT SET and p-CLIQUE are $W[1]$-complete*

$W[t]$-Completeness

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Theorem

- *p-INDEPENDENT SET and p-CLIQUE are $W[1]$-complete*
- *p-DOMINATING SET is $W[2]$-complete*

Theorem

For every $t \geq 1$, $W[t] = \mathcal{FPT}$ if and only if a $W[t]$-hard problem is a member of $\mathcal{FPT}$.

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Let (Π, κ) be a parameterized problem with $\mathcal{I}$ the set of instances of Π. A in polynomial time computable function $f : \mathcal{I} \times \mathbb{N} \to \mathcal{I} \times \mathbb{N}$ is called kernelization for (Π, κ) if $(I', \kappa(I')) = f(I, \kappa(I))$ satisfies the following three properties:

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2. There exists a function $f' : \mathbb{N} \to \mathbb{N}$ such that $|I'| \leq f'(\kappa(I))$

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3. $\kappa(I') \leq \kappa(I)$

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Let (Π, κ) be a parameterized problem with $\mathcal{I}$ the set of instances of Π. A in polynomial time computable function $f : \mathcal{I} \times \mathbb{N} \to \mathcal{I} \times \mathbb{N}$ is called kernelization for (Π, κ) if $(I', \kappa(I')) = f(I, \kappa(I))$ satisfies the following three properties:

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2. There exists a function $f' : \mathbb{N} \to \mathbb{N}$ such that $|I'| \leq f'(\kappa(I))$
3. $\kappa(I') \leq \kappa(I)$

The instance I' is called kernel of (Π, κ) and $f'(\kappa(I))$ is called the size of the kernel.

Idea: reduce an instance I to an instance I' which size only depends on the parameter, not on the original instance size

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Example: p-VERTEX COVER

2 Reduction rules:

Lemma

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Let $G = (V, E)$ be a graph without isolated vertices. If G has a vertex cover of size at most k and $\Delta(G) \leq d$, then G has at most $k(d + 1)$ vertices.

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p-VERTEX COVER has a kernel of size at most $k(k+1)$, where k is the parameter of the problem.

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Corollary

p-VERTEX COVER$\in$ FPT

FPT algorithm for p-VERTEX COVER

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Theorem

The FPT-algorithm for p-VERTEX COVER decides for every graph G with n vertices in $O(kn + k^{2k})$ whether G has a vertex cover of size at most k.

Lemma

Let $G = (V, E)$ *be a graph and* $v \in V$ *a vertex of degree 1 with neighbor* w*.* G *has a vertex cover of size* k *if and only if* $G - \{v, w\}$ *has a vertex cover of size* $k - 1$*.*

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The search tree defined by VC1, VC2, and VC3 for p-VERTEX COVER has a size of $O(1.47^k)$.

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A combination of kernelization and search tree is also possible.

Algorithmic Graph Theory: How hard is your combinatorial optimization problem?

Arie M.C.A. Koster

Lecture 9

Clemson, June 13, 2017

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