Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
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16 Improper Integrals

In the previous chapter the integral $\int_a^b f(x)dx$ is defined over a finite interval $[a, b]$. The function f was assumed to be continuous or at least bounded, otherwise the integral was not guaranteed to exist. In this section we investigate what happens when these conditions are not met.

Definition 1. *An integral is an* improper integral *if either the interval of integration is not finite (type I) or if the function to integrate is not continuous in the interval of integration (type II).*

Example

i)
$$\int_0^\infty e^{-x}\,dx \quad \text{is an improper integral of type I.}$$

ii)
$$\int_0^1 \frac{1}{x}\,dx \quad \text{is an improper integral of type II.}$$

Definition 2. *Improper integrals of type I are evaluated as follows: If $\int_a^t f(x)\,dx$ exists for all $t \geq a$, then we define*

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided the limit exists as a finite number . In this case, $\int_a^\infty f(x)\,dx$ is convergent. Otherwise it is divergent. If the lower bound is not finite, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx.$$

If both bounds are not finite, then we split the integral

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

The integrals on the right side are evaluated as shown before.

Definition 3. *Improper integrals of type II are evaluated as follows: If f is continuous on $[a, b)$ and not continuous at b then we define*

$$\int_a^b f(x)\,dx = \lim_{t\to b}\int_a^t f(x)\,dx$$

provided the limit exists as a finite number . In this case, $\int_a^b f(x)\,dx$ is convergent. Otherwise it is divergent. We get the analog form, if f is not continuous at a.

Evaluating an improper integral is really two problems. It is an integral problem and a limit problem. It is best to do them separately.

Example

i) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

   This is an improper integral of type I. We evaluate it by finding $\displaystyle\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$.

$$\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^t = -\frac{1}{t} + 1$$

$$\Rightarrow \quad \lim_{t\to\infty} -\frac{1}{t} + 1 = 1 \quad \text{hence} \quad \int_1^\infty \frac{1}{x^2}\,dx = 1.$$

ii) $\displaystyle\int_0^1 \frac{1}{x}\,dx$

   This is an improper integral of type II. f is not continuous at $x = 0$.

$$\int_t^1 \frac{1}{x}\,dx = \ln(x)\bigg|_t^1 = -\ln(t)$$

$$\Rightarrow \quad \lim_{t\to 0} -\ln(t) = \infty \quad \text{hence} \quad \int_0^1 \frac{1}{x}\,dx = \infty.$$

Theorem 4 (Comparison Theorem)**.** *Suppose that f and g are two continuous functions for $x \geq a$ such that $0 \leq g(x) \leq f(x)$. Then the following is true:*

i) If $\displaystyle\int_a^\infty f(x)\,dx$ converges then $\displaystyle\int_a^\infty g(x)\,dx$ also converges.

ii) If $\displaystyle\int_a^\infty g(x)\,dx$ diverges then $\displaystyle\int_a^\infty f(x)\,dx$ also diverges.