11 Continuity

Definition 1 (limits of functions via sequences). *Let $f\colon D \to \mathbb{R}$ be a function, where $D \subset \mathbb{R}^n$, and $x_0 \in \mathbb{R}^n$. A number $a \in \mathbb{R}$ is called* limit of f in x_0 *if for all sequences $(y_n) \subset D \setminus \{x_0\}$ satisfying $y_n \to x_0$:*

$$\lim_{n\to\infty} f(y_n) = a.$$

We write $f(x) \to a$ for $x \to x_0$.

Definition 2 (continuity). *Let $f\colon D \to \mathbb{R}$ be a function, where $D \subset \mathbb{R}^n$. The function f is called*

(i) continuous at $x_0 \in D$ *if*

$$f(x_0) = \lim_{x\to x_0} f(x).$$

(ii) continuous on D *if f is continuous at every point $x_0 \in D$.*

Theorem 3 (ε-δ-criterion). *Let $f\colon D \to \mathbb{R}$ be a function, where $D \subset \mathbb{R}^n$. The function f is continuous at x_0 if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\text{for all } x \in D\colon |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Theorem 4 (continuity & operations on functions). *If $f, g\colon D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$, are functions that are continuous at x_0, then $f + g$, $f \cdot g$, and $\frac{f}{g}$ (for $g(x_0) \neq 0$) are continuous at x_0.*

Theorem 5 (continuity & concatenation of functions). *If $f\colon D_1 \to D_2$ and $g\colon D_2 \to \mathbb{R}$, where $D_1, D_2 \subset \mathbb{R}$, are continuous at x_0 and $f(x_0)$, respectively, then $g \circ f\colon D_1 \to \mathbb{R}$ is continuous at x_0.*

Theorem 6 (continuity of f^{-1}). *Let $f\colon I \to \mathbb{R}$ be a continuous function, where I is an interval. If f is strictly monotonic on I, then $f^{-1}\colon f(I) \to I$ is continuous on $f(I)$.*

Theorem 7 (classes of continuous functions). *The following functions are continuous on their respective domains.*

(i) The absolute value $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(\mathbf{x}) = |\mathbf{x}|$;

(ii) Polynomials;

(iii) Roots;

(iv) Trigonometric functions;

(v) Exponential functions $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = c^x$ *for fixed* $c > 0$;

(vi) The minimum and maximum function

$$\begin{aligned} \min(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R}, \min(\mathbf{x}) &= \min\{x_1, x_2, \dots, x_n\}, \\ \max(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R}, \max(\mathbf{x}) &= \max\{x_1, x_2, \dots, x_n\}. \end{aligned}$$

Theorem 8 (intervals & continuous functions). *Let* $f\colon D \to \mathbb{R}$ *be a continuous function, where* $D \subset \mathbb{R}$. *If* $[a,b] \subset D$, *there exist* $c, d \in \mathbb{R}$ *such that* $f([a,b]) = [c,d]$, *i.e. the image of a closed interval under a continuous function is again a closed interval.*

Corollary 9 (Weierstrass' extremal value theorem). *Let* $f\colon D \to \mathbb{R}$ *be a continuous function, where* $D \subset \mathbb{R}$. *If* $[a,b] \subset D$, *then* f *has a minimum and maximum in* $[a,b]$, *i.e. there exist* $x_{min}, x_{max} \in [a,b]$ *such that* $f(x_{min}) \leq f(x) \leq f(x_{max})$ *for every* $x \in [a,b]$.

Corollary 10 (Intermediate value theorem). *Let* $f\colon D \to \mathbb{R}$ *be a continuous function, where* $D \subset \mathbb{R}$, *and* $[a,b] \subset D$. *For every* y *between* $f(a)$ *and* $f(b)$ *there is an* $x \in [a,b]$ *with* $f(x) = y$.

Corollary 11 (existence of roots). *Let* $f\colon D \to \mathbb{R}$ *be a continuous function, where* $D \subset \mathbb{R}$, *and* $[a,b] \subset D$. *If* $f(a)f(b) < 0$, *there is an* $x \in [a,b]$ *with* $f(x) = 0$.