Lehrstuhl II für Mathematik http://www.math2.rwth-aachen.de
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9 Sequences

Definition 1 (sequence). *A* sequence *of numbers is a mapping*

$$x\colon \mathbb{N} \to \mathbb{R}, \\ n \mapsto x_n,$$

i.e. a rule assigning each natural number a real number x_n*. We also use the notations* $(x_n)_{n\in\mathbb{N}}$ *or* (x_n) *or* $x_1, x_2, x_3, \ldots$*. Instead of* $\mathbb{N}$ *the set* $\mathbb{N}_0$ *may be used at times.*

Definition 2 (convergence). *A sequence* (x_n) converges *to a* limit $x \in \mathbb{R}$ *if for every* $\varepsilon > 0$ *there is a number* $n_0 \in \mathbb{N}$ *such that*

$$\text{for all } n \geq n_0\colon |x_n - x| < \varepsilon.$$

We write

$$x = \lim_{n\to\infty} x_n \text{ or } x_n \to x.$$

A sequence that does not converge is divergent*.*

Theorem 3 (uniqueness of limits). *A sequence has at most one limit.*

Theorem 4 (Sandwich Theorem). *Let* $(x_n), (y_n), (z_n)$ *be sequences such that* $x_n \to x$ *and* $z_n \to x$*. If there exists a number* $n_0 \in \mathbb{N}$ *such that*

$$\text{for all } n \geq n_0\colon x_n \leq y_n \leq z_n,$$

then $y_n \to x$*.*

Theorem 5 (convergence & boundedness). *If a sequence converges, then it is bounded.*

Theorem 6 (monotonicity, boundedness & convergence). *If a sequence is monotonic and bounded, then it is convergent.*

Definition 7 (indefinite convergence). *Let* (x_n) *be a sequence. If for every* $b \in \mathbb{R}$*, there is a number* $n_0 \in \mathbb{N}$ *such that*

$$\text{for all } n \geq n_0\colon x_n \geq b, \\ [\text{for all } n \geq n_0\colon x_n \leq b,]$$

we write $x_n \to \infty$ *[*$x_n \to -\infty$*].*

Theorem 8 (computations with limits). *If $(x_n), (y_n)$ are sequences such that $x_n \to x$ and $y_n \to y$, then*

(i) $\lim_{n\to\infty}(x_n + y_n) = x + y$.

(ii) $\lim_{n\to\infty}(x_n \cdot y_n) = xy$ *(in particular* $\lim_{n\to\infty}(c \cdot x_n) = cx$ *for* $c \in \mathbb{R}$*).*

(iii) *for* $y \neq 0$: $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

(iv) $\lim_{n\to\infty}|x_n| = |x|$

Corollary 9 (limits of ratios of polynomials). *Let P, Q be polynomials, i.e.*

$$P(x) = \sum_{i=0}^{m} a_i x^i \text{ and } Q(x) = \sum_{i=0}^{k} b_i x^i$$

with $m, k \in \mathbb{N}$, and $a_r, b_s \in \mathbb{R}$ for $r = 0, 1, \ldots, m$ and $s = 0, 1, \ldots, k$, and $a_m, b_k \neq 0$. Then

$$\lim_{n\to\infty} \frac{P(n)}{Q(n)} = \begin{cases} \infty & \text{if } m > k \text{ and } a_m b_k > 0, \\ -\infty & \text{if } m > k \text{ and } a_m b_k < 0, \\ \frac{a_m}{b_k} & \text{if } m = k, \\ 0 & \text{if } m < k \end{cases}.$$

Definition 10 (asymtotic growth of sequences). *Let $(x_n), (y_n)$ be sequences with $x_n \to \infty$ and $y_n \to \infty$. The sequence y_n* grows faster *to ∞ than (x_n) if*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0.$$

Theorem 11 (speed of convergence). *In the following list each sequence grows faster to ∞ than the precedent sequence.*

(i) (n^k) *for* $k \in \mathbb{N}$,

(ii) (q^n) *for* $q > 1$,

(iii) $(n!)$,

(iv) (n^n),

(v) (2^{n^2}).

Theorem 12 (Cauchy criterion). *A sequence (x_n) converges if and only if for every $\varepsilon > 0$ there is a number $n_0 \in \mathbb{N}$ such that*

$$\text{for all } m, n \geq n_0 \colon |x_n - x_m| < \varepsilon.$$