7 Eigenvectors and eigenvalues

Definition 1 (Eigenvectors and Eigenvalues). *Let $A \in \mathbb{R}^{n\times n}$ be a matrix. A vector $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^n$ is called* eigenvector *of A if*

$$A\mathbf{v} = \lambda\mathbf{v},$$

for $\lambda \in \mathbb{R}$. λ is called eigenvalue *of the matrix A.*

Definition 2 (Eigenspace). *Let $A \in \mathbb{R}^{n\times n}$ be a matrix with an eigenvalue $\lambda \in \mathbb{R}$. The set*

$$V_\lambda = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \lambda\mathbf{v}\}$$

is called eigenspace *of λ. It is a linear subspace of $\mathbb{R}^n$.*

We can rewrite the condition $A\mathbf{v} = \lambda\mathbf{v}$ as

$$(A - \lambda I_n)\mathbf{v} = 0.$$

To find eigenvectors $\mathbf{v}$ corresponding to an eigenvalue λ, we solve the homogenous system of linear equations given by

$$(A - \lambda I_n | 0).$$

Theorem 3. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix with pairwise distinct eigenvalues $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ and eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_m$ such that $v_i \in V_{\lambda_i}$. Then the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_m$ are linearly independent.*

Definition 4 (Characteristic polynomial). *Let $A \in \mathbb{R}^{n\times n}$ be a matrix. The polynomial*

$$\det(A - \lambda I_n)$$

is called characteristic polynomial *of A. It is an n-th order polynomial with variable λ.*

Theorem 5. *Let $A \in \mathbb{R}^{n\times n}$ and $p(\lambda) = \det(A - \lambda I_n)$ its characteristic polynomial. The eigenvalues of A are exactly the zeros of p.*

Theorem 6 (The Diagonalization Theorem)**.** *Let* $A \in \mathbb{R}^{n\times n}$*. If* $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ *are linearly independent eigenvectors of* A *and* $\lambda_1, \lambda_2, \dots, \lambda_n$ *are their corresponding eigenvalues, then*

$$V^{-1}AV = D,$$

where

$$V = (\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}) \in \mathbb{R}^{n\times n}$$

and D *is the diagonal matrix*

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$