Lehrstuhl II für Mathematik
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Calculus and Linear Algebra

Exam

Exercise 1: Let A, b be defined by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & t & -4 \\ -3 & 4 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For which values of t does $Ax = b$ have a unique solution?

Solution: The system $Ax = b$ is uniquely solvable precisely if $\det A \neq 0$. Since

$$\begin{aligned} \det A =& 1 \cdot t \cdot (-1) + 2 \cdot (-4) \cdot (-3) + 1 \cdot (-2) \cdot 4 \\ & - 1 \cdot t \cdot (-3) - (-4) \cdot 4 \cdot 1 - (-1) \cdot 2 \cdot (-2) \\ =& 2t + 28, \end{aligned}$$

$Ax = b$ has a unique solution if and only if $t \neq -14$.

Exercise 2: Let

$$L = \left\{ x \in \mathbb{R}^3 : x = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + r \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

be a straight line and

$$P = \left\{ x \in \mathbb{R}^3 : x = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right\}$$

a plane in $\mathbb{R}^3$. If L and P intersect, compute the intersection set $L \cap P$. Otherwise, compute the distance between them.

Solution: A vector $x \in \mathbb{R}^3$ is in $L \cap P$ if and only if there exist $r, s, t \in \mathbb{R}$ such that

$$\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + r \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = x = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}.$$

This equation is equivalent to

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 1 \\ 0 & -1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 2 & -3 & 3 \\ 0 & -1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 2 & -3 & 3 \\ 0 & 0 & 3 & 3 \end{array}\right).$$

Hence $x_3 = 1$, $x_2 = \frac{1}{2}(3 + 3x_1) = 3$ and $x_1 = 4 - x_2 + x_3 = 2$ and thus,

$$L \cap P = \left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Exercise 3: Let (x_n) be a sequence that is recursively defined by

$$x_1 = 2, \quad x_{n+1} = \frac{2x_n}{2+x_n} + 2 \ (n \in \mathbb{N}).$$

Show that (x_n) is converging and compute its limit x.

Solution: (i) *The sequence is bounded from above.* The proof is by induction on n. Obviously, $x_1 = 2 \le 1 + \sqrt{5}$. Furthermore,

$$\begin{aligned} x_{n+1} = \frac{2x_n}{2+x_n} + 2 &\le 1 + \sqrt{5} \\ \iff \frac{2x_n}{2+x_n} &\le \sqrt{5} - 1 \\ \iff 2x_n &\le 2(\sqrt{5}-1) + x_n(\sqrt{5}-1) \\ \iff (3-\sqrt{5})x_n &\le 2(\sqrt{5}-1). \end{aligned}$$

Since $x_n \le 1 + \sqrt{5}$, it follows that

$$(3-\sqrt{5})x_n \le (3-\sqrt{5})(1+\sqrt{5}) = 2(2\sqrt{5}-1).$$

(ii) *The sequence is monotonically increasing,* since

$$\begin{aligned} x_{n+1} - x_n = \frac{2x_n}{2+x_n} + 2 - x_n &= -\frac{x_n^2 - 2x_n - 4}{2+x_n} \ge 0 \\ &\iff x_n^2 - 2x_n - 4 \le 0 \\ &\iff (x_n - 1)^2 \le 5. \end{aligned}$$

In view of (i) the last inequality is true.

(iii) Because of the observations above, the sequence is convergent. From the defining equation it follows that

$$x \leftarrow x_{n+1} = \frac{2x_n}{2+x_n} + 2 \to \frac{2x}{2+x} + 2$$

as $x \to \infty$. Solving the above equation for x yields

$$\begin{aligned} x = \frac{2x}{2+x} + 2 &\iff x(2+x) = 2x + 2(2+x) \\ &\iff x^2 - 2x + 1 = 5 \\ &\iff (x-1)^2 = 5 \\ &\iff x = 1+\sqrt{5} \ \vee \ x = 1 - \sqrt{5}. \end{aligned}$$

Since $x_n \ge 0$ for all $n \in \mathbb{N}$, it follows that $x = 1 + \sqrt{5}$.

Exercise 4: Determine whether the series $\sum_{n=1}^{\infty} x_n$ is convergent or not.

a) $x_n = \dfrac{1}{(\log(n+1))^n}$; b) $x_n = \dfrac{(n+1)^n}{n^{n+1}}$.

Solution: a) Since $\sqrt[n]{x_n} = 1/\log(n+1) \to 0$ as $n \to \infty$, the series is convergent in view of the root criterion.

b) Since

$$\frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\left(\frac{n+1}{n}\right)^n \geq \frac{1}{n},$$

the series is divergent in view of the comparison criterion.

Exercise 5: For which values $x \in [-\pi, \pi]$ is

$$f(x) = \frac{\sqrt{\log(1 + \cos x))}}{x}$$

defined?

Solution: Because of the denominator the quotient is not defined in $x = 0$. Since the square root is only defined for arguments ≥ 0, we have to answer where $\log(1+\cos x) \geq 0$ is true. This boils down to

$$\cos x \geq 0 \overset{x\in[-\pi,\pi]}{\iff} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

All in all the answer is $x \in [-\pi/2, 0) \cup (0, \pi/2]$.

Exercise 6: Let the function f be defined by

$$f(x) = \frac{|x-1|(2x+1)\sin(\frac{\pi x}{2})}{x(2x+1)(x-1)(x+1)}.$$

Compute the domain of continuity of f. At which points is f continuously extendable?

Solution: The maximal domain of definition is $\mathbb{R}\setminus\{-1,-\frac{1}{2},0,1\}$ and as a composition of continuous functions, f is continuous on its domain. To decide where f is continuously extendable, we have to take a closer look at -1, $-\frac{1}{2}$, 0 and 1. Since $\frac{|x|}{x}=1$ for $x>0$ and $\frac{|x|}{x}=-1$ for $x<0$, we conclude that

$$f(x) = \begin{cases} \dfrac{\sin(\frac{\pi x}{2})}{x(x+1)}, & x>1 \\ -\dfrac{\sin(\frac{\pi x}{2})}{x(x+1)}, & x<1 \text{ and } x\neq -\frac{1}{2} \end{cases}.$$

If $x\to -1$, then $\sin(\frac{\pi x}{2})\to -1$ and $x(x+1)\to 0$. Hence f is divergent in -1. For the second point we have

$$\lim_{x\to-\frac{1}{2}} f(x) = \lim_{x\to-\frac{1}{2}} -\frac{\sin(\frac{\pi x}{2})}{x(x+1)} = -\lim_{x\to-\frac{1}{2}} \frac{\sin(\frac{\pi x}{2})}{x(x+1)} = -\frac{-\sqrt{2}/2}{1/4} = 2\sqrt{2}.$$

Now recall that

$$\lim_{x\to 0}\frac{\sin(cx)}{x} = 1$$

for every $c\in\mathbb{R}$. It follows that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} -\frac{\sin(\frac{\pi x}{2})}{x(x+1)} = -\frac{1}{2}\lim_{x\to 0}\frac{\sin(\frac{\pi x}{2})}{\frac{x}{2}}\lim_{x\to 0}\frac{1}{x+1} = -\frac{1}{2}.$$

For the last point note that if $x>1$, then

$$f(x) = \frac{\sin(\frac{\pi x}{2})}{x(x+1)} \to \frac{1}{2} \quad (x\to 1)$$

and if $x<1$, then

$$f(x) = -\frac{\sin(\frac{\pi x}{2})}{x(x+1)} \to -\frac{1}{2} \quad (x\to 1)$$

Therefore the limit of f in 1 does not exist. All in all, f is continuously extendable by $f(x)=2\sqrt{2}$ in $x=-\frac{1}{2}$, and by $f(x)=-\frac{1}{2}$ in $x=0$.

Exercise 7: Compute all extrema and all inflection points of

$$f(x) = xe^{1-2x^2}$$

on the interval $[-2, 8)$.

Solution: We compute the first and second derivative of f, since we shall need them later. It holds

$$f'(x) = 1 \cdot e^{1-2x^2} + xe^{1-2x^2}(-4x) = (1-4x^2)e^{1-2x^2},$$
$$f''(x) = -8xe^{1-2x^2} + (1-4x^2)e^{1-2x^2}(-4x) = (16x^3 - 12x)e^{1-2x^2}.$$

A necessary criterion for a point x to be a local extremum of f on the *open* interval $(-2, 8)$ is $f'(x) = 0$. We have

$$(1-4x^2)e^{1-2x^2} = f'(x) = 0 \iff 1 - 4x^2 = 0 \iff x = -\frac{1}{2} \vee x = \frac{1}{2}.$$

A sufficient criterion for a point x to be a local extremum of f on the *open* interval $(-2, 8)$ is $f''(x) \neq 0$. Since

$$-f''(-\frac{1}{2}) = f''(\frac{1}{2}) = (\frac{16}{2^3} - \frac{12}{2})e^{1-2(\frac{1}{2})^2} = -4e^{\frac{1}{2}} < 0,$$

the function f has a local minimum in $x = -1/2$ and a local maximum in $x = 1/2$. It remains to check the behaviour of f at the boundaries of $[-2, 8)$. Since

$$f'(x) \begin{cases} < 0, & x < -\frac{1}{2} \\ > 0, & -\frac{1}{2} < x < \frac{1}{2} \\ < 0, & \frac{1}{2} < x \end{cases},$$

the point $x = -2$ is a local maximum of f. In addition,

$$f(-2) = -2e^{-7} < 0 < \frac{1}{2}e^{\frac{1}{2}} = f(\frac{1}{2}) \text{ and } f(-\frac{1}{2}) = -\frac{1}{2}e^{\frac{1}{2}} < 0 < 8e^{-127} = \lim_{x \to 8} f(x)$$

and therefore $x = \frac{1}{2}$ is the global maximum, and $x = -\frac{1}{2}$ is the global minimum of f on $[-2, 8)$.

A sufficient criterion for a point x to be an inflection of f is $f'(x) = 0$. We have

$$\begin{aligned} &(16x^3 - 12x)e^{1-2x^2} = f''(x) = 0 \\ \iff\ & 16x^3 - 12x = 16x(x - \frac{\sqrt{3}}{2})(x + \frac{\sqrt{3}}{2}) = 0 \\ \iff\ & x = 0 \vee x = -\frac{\sqrt{3}}{2} \vee x = \frac{\sqrt{3}}{2}. \end{aligned}$$

Since

$$f''(x) \begin{cases} < 0, & x \in (-\infty, -\frac{\sqrt{3}}{2}) \\ > 0, & x \in (-\frac{\sqrt{3}}{2}, 0) \\ < 0, & x \in (0, \frac{\sqrt{3}}{2}) \\ > 0, & x \in (\frac{\sqrt{3}}{2}, \infty) \end{cases},$$

the function f changes its behaviour from concavity to convexity in $x = -\sqrt{3}/2$ and $x = \sqrt{3}/2$, and from convexity to concavity in $x = 0$.

Exercise 8: Let $f(x) = \frac{\log x}{x}$. Compute the TAYLOR polynomial $T_2^1 f(x)$ of second order of f around $x_0 = 1$ in the form

$$T_2^1 f(x) = a_0 + a_1 x + a_2 x^2,$$

where $a_0, a_1, a_2 \in \mathbb{R}$.

Solution: We compute the first and second derivative of f, since we shall need them later. It holds

$$f'(x) = \frac{\frac{1}{x}x - \log x}{x^2} = \frac{1 - \log x}{x^2},$$
$$f''(x) = \frac{-\frac{1}{x}x^2 - (1 - \log x)2x}{x^4} = \frac{-3 + 2\log x}{x^3}.$$

It follows that

$$\begin{aligned} T_2^1 f(x) &= \sum_{k=0}^{2} \frac{f^{(k)}(1)}{k!}(x-1)^k \\ &= f(1) + f'(1)(x-1) + \frac{1}{2} f''(1)(x-1)^2 \\ &= 0 + 1 \cdot (x-1) - \frac{3}{2}(x-1)^2 \\ &= -\frac{5}{2} + 4x - \frac{3}{2}x^2. \end{aligned}$$

Exercise 9: Let

$$M = \left\{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1, 0 \le x \le \frac{y}{2}\right\}$$
$$= \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le \frac{1}{2}, 2x \le y \le 1\right\}.$$

Compute

$$\iint_M y^2 \sin\left(\frac{2\pi x}{y}\right) \mathrm{d}(x,y).$$

Solution: It holds

$$\begin{aligned}
\iint_M y^2 \sin\left(\frac{2\pi x}{y}\right) \mathrm{d}(x,y) &= \int_0^1 \int_0^{\frac{y}{2}} y^2 \sin\left(\frac{2\pi x}{y}\right) \mathrm{d}x \, \mathrm{d}y \\
&= -\int_0^1 \frac{y^3}{2\pi} \cos\left(\frac{2\pi x}{y}\right)\Big|_0^{\frac{y}{2}} \mathrm{d}y \\
&= \int_0^1 \frac{y^3}{\pi} \mathrm{d}y \\
&= \frac{y^4}{4\pi}\Big|_0^1 \\
&= \frac{1}{4\pi}.
\end{aligned}$$

Exercise 10: Determine

$$\int_{\log 3}^{\log 8} \frac{1}{\sqrt{1+e^x}}\,\mathrm{d}x.$$

Solution: We substitute

$$\sqrt{1+e^x} = u, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{e^x}{2\sqrt{1+e^x}}, \qquad \mathrm{d}x = \frac{2u}{u^2-1}\,\mathrm{d}u.$$

It follows that

$$\int \frac{1}{\sqrt{1+e^x}}\,\mathrm{d}x = \int \frac{2}{u^2-1}\,\mathrm{d}u = \int \frac{1}{u-1} - \frac{1}{u+1}\,\mathrm{d}u$$
$$= \log(u-1) - \log(u+1) = \log\frac{u-1}{u+1} = \log\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}.$$

Plugging in the boundaries we get

$$\int_{\log 3}^{\log 8} \frac{1}{\sqrt{1+e^x}}\,\mathrm{d}x = \log\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\bigg|_{\log 3}^{\log 8} = \log\frac{1}{2} - \log\frac{1}{3} = \log\frac{3}{2}.$$

Exercise 11: Determine and sketch the set

$$\{z \in \mathbb{C}\colon\ |z|^2 \leq 2\,\mathrm{Im}(z) + 3\}.$$

Solution: Let $z = x + iy$. It holds

$$\begin{aligned} &|z|^2 \leq 2\,\mathrm{Im}(z) + 3 \\ \Longleftrightarrow\ &x^2 + y^2 \leq 2y + 3 \\ \Longleftrightarrow\ &x^2 + y^2 - 2y + 1 \leq 4 \\ \Longleftrightarrow\ &x^2 + (y-1)^2 \leq 2^2. \end{aligned}$$

Hence the set describes a disc with radius 2 around the point i (including the boundary).

Exercise 12: Determine the real and imaginary part of

$$z = \left(\frac{2+3i}{1-2i} + \frac{i}{3+i}\right)^{-1}.$$

Solution: It holds

$$\begin{aligned}\left(\frac{2+3i}{1-2i} + \frac{i}{3+i}\right)^{-1} &= \left(\frac{(2+3i)(3+i)+i(1-2i)}{(1-2i)(3+i)}\right)^{-1} \\ &= \left(\frac{6+11i+3i^2+i-2i^2}{3-5i-2i^2}\right)^{-1} = \frac{5-5i}{5+12i} \\ &= \frac{(5-5i)(5-12i)}{(5+12i)(5-12i)} = \frac{25-85i+60i^2}{25-144i^2} = -\frac{35}{169} - \frac{85}{169}i.\end{aligned}$$

Hence $\mathrm{Re}(z) = -35/169$ and $\mathrm{Im}(z) = -85/169$.