Lehrstuhl A für Mathematik
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Aachen, 13 March 2009

Calculus and Linear Algebra

Solutions of the exam

Exercise 1 (10 points).

a) Determine and sketch the set

$$M = \left\{ z \in \mathbb{C} : \operatorname{Re}\left(\frac{i}{z}\right) < 0 \right\}.$$

b) Determine the real and imaginary part of

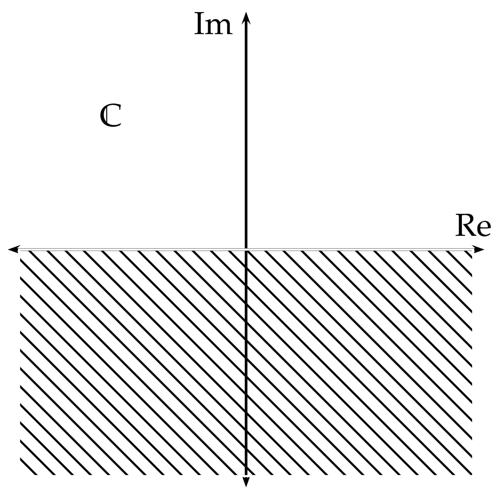
$$\left(\frac{1-2i}{8-i}\right)^{-1}.$$

Solution:

a) Let $z = x + iy$. Then

$$\operatorname{Re}\left(\frac{i}{z}\right) = \operatorname{Re}\left(\frac{i\bar{z}}{z\bar{z}}\right) = \operatorname{Re}\left(\frac{ix+y}{x^2+y^2}\right) = \frac{y}{x^2+y^2}.$$

The last expression is negative if and only if $y < 0$.
Sketch:



b) We have

$$\left(\frac{1-2i}{8-i}\right)^{-1} = \frac{8-i}{1-2i} = \frac{(8-i)(1+2i)}{(1-2i)(1+2i)} = \frac{10+15i}{5} = 2+3i.$$

Exercise 2 (13 points).

Let

$$A_t = \begin{pmatrix} t-1 & 3 & 1 \\ 0 & t+2 & 1 \\ -1 & 2 & 1 \end{pmatrix},$$

where $t \in \mathbb{R}$.

a) For which values of t is A_t invertible?

b) Solve the system of linear equations

$$A_0 x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution:

a) Since

$$\begin{aligned} \det(A_t) &= (t-1)\cdot(t+2)\cdot 1 + 3\cdot 1\cdot(-1) + 1\cdot 0\cdot 2 \\ &\quad - 1\cdot(t+2)\cdot(-1) - 1\cdot 2\cdot(t-1) - 3\cdot 0\cdot 1 \\ &= t^2 - 1 \\ &= (t+1)(t-1), \end{aligned}$$

the matrix A_t is invertible for $t \in \mathbb{R} \setminus \{-1, 1\}$.

b) Using Gauss' algorithm, we get

$$A_0 x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} -1 & 3 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ -1 & 2 & 1 & 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & -3 & -1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

Hence $x_2 = 0$, $x_3 = 1 - 2x_2 = 1$ and $x_1 = -1 + 3x_2 + x_3 = 0$.

Exercise 3 (12 points).

Let the function f be defined by

$$f(x) = \frac{(x^2-1)\sin x}{x^3+x^2-2x}.$$

a) Determine the domain of f.

b) Determine the domain of continuity of f. At which points exists a continuous extension of f?

Solution:

a) The domain of f consists of all points $x \in \mathbb{R}$ at which the denominator does not vanish. We have

$$x^3 + x^2 - 2x = x(x^2+x-2) = x(x+2)(x-1)$$

and therefore $D(f) = \mathbb{R} \setminus \{-2, 0, 1\}$.

b) The function f is continuous on its domain as a composition of continuous functions (polynomials, sin, multiplication, division). Note that

$$f(x) = \frac{(x+1)(x-1)\sin x}{x(x+2)(x-1)}.$$

Hence

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x+1)\sin x}{x(x+2)} = \frac{2}{3}\sin 1$$

and

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\xrightarrow{x\to 0} 1} \frac{x+1}{x+2} = \frac{1}{2}.$$

Thus f is extendable in 0 and 1. Furthermore,

$$\lim_{x\to -2} (x+1)(x-1)\sin x = 3\sin(-2)$$

and

$$\lim_{x\to -2} x(x+2)(x-1) = 0.$$

Therefore, f is not continuously extendable in $x = -2$.

Exercise 4 (13 points).

Let the function f be defined by

$$f(x) = \frac{x+1}{x^2+15}.$$

a) Determine the monotonicity intervals of f.

b) Determine all extremal points of f on $[0,4]$.

Solution: First we compute the derivative of f as we need it for both a) and b):

$$\begin{aligned} f'(x) &= \frac{(x+1)'(x^2+15) - (x+1)(x^2+15)'}{(x^2+15)^2} \\ &= \frac{x^2+15-2x(x+1)}{(x^2+15)^2} \\ &= \frac{-x^2-2x+15}{(x^2+15)^2} \\ &= \frac{-(x-3)(x+5)}{(x^2+15)^2}. \end{aligned}$$

a) The monotonicity intervals of f are determined by the sign of f'. Note that the sign of f' only depends on the sign of the enumerator, since the denominator is a square. More precisely,

$$-(x-3)(x+5) \begin{cases} < 0 & x \in (-\infty,-5) \cup (3,\infty) \\ = 0 & x \in \{-5,3\} \\ > 0 & x \in (-5,3) \end{cases} .$$

Hence f is increasing on $[-5,3]$ and decreasing on $(-\infty,-5] \cup [3,\infty)$.

b) We consider f' on the interval $[0,4]$. Since

$$f'(x) \begin{cases} < 0 & x \in (3,4] \\ = 0 & x = 3 \\ > 0 & x \in [0,3) \end{cases} ,$$

the function f has a global maximum at $x = 3$. Furthermore, $x = 0$ and $x = 4$ are minimal points of f. Since $f(0) = \frac{1}{15} < \frac{5}{31} = f(4)$, the first one is a global minimum and the latter one is a local minimum.

Exercise 5 (10 points).

Determine whether the following sequences $(a_n)_{n\in\mathbb{N}}$ are convergent or not, and compute the limits if possible.

a) $a_n = \left(\dfrac{2n-1}{2n}\right)^{6n}$.

b) $a_n = \dfrac{\sqrt{n^4-2n^3+1}-n^2}{4n}$.

Solution:

a) Observe that

$$\left(\frac{2n-1}{2n}\right)^{6n} = \left[\left(\frac{2n}{2n-1}\right)^{6n}\right]^{-1} = \left\{\left[\underbrace{\left(1+\frac{1}{2n-1}\right)^{2n-1}}_{\to e}\right]^3 \underbrace{\left[1+\frac{1}{2n-1}\right]^3}_{\to 1}\right\}^{-1}.$$

Hence

$$\lim_{n\to\infty}\left(\frac{2n-1}{2n}\right)^{6n} = e^{-3}.$$

b) We have

$$\begin{aligned}\frac{\sqrt{n^4-2n^3+1}-n^2}{4n} &= \frac{(\sqrt{n^4-2n^3+1}-n^2)(\sqrt{n^4-2n^3+1}+n^2)}{4n(\sqrt{n^4-2n^3+1}+n^2)}\\ &= \frac{(n^4-2n^3+1)-n^4}{4n(\sqrt{n^4-2n^3+1}+n^2)}\\ &= \frac{-2n^3+1}{4n(\sqrt{n^4-2n^3+1}+n^2)}\\ &= \frac{-2+\frac{1}{n^3}}{4(\sqrt{1-\frac{2}{n}+\frac{1}{n^4}}+1)}\\ &\overset{n\to\infty}{\longrightarrow} -\frac{1}{4}.\end{aligned}$$

Exercise 6 (12 points).

Determine whether the following series $\sum_{n=1}^{\infty} a_n$ are convergent or not.

a) $a_n = \dfrac{2^n}{n!}$,

b) $a_n = \dfrac{\sqrt{n^2-1}}{n^2-2}$.

Solution:

a) We use the quotient criterion:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \\ &= \frac{n!2^{n+1}}{(n+1)!2^n} \\ &= \frac{2}{n+1} \\ &\overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore, the series is convergent.

b) We use the majorant criterion:

$$\begin{aligned}\left|\frac{\sqrt{n^2-1}}{n^2-2}\right| &= \frac{\sqrt{n^2-1}}{n^2-2} \\ &\geqslant \frac{\sqrt{(n-1)^2}}{n^2-2} \\ &= \frac{n-1}{n^2-2}.\end{aligned}$$

If we suppose $n \geqslant 2$, we can go on as follows:

$$\begin{aligned}\frac{n-1}{n^2-2} &\geqslant \frac{n-1}{n^2-1} \\ &= \frac{n-1}{(n-1)(n+1)} \\ &= \frac{1}{n+1}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$ diverges, it follows that $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^2-2}$ diverges.

Exercise 7 (10 points).

Let $f(x) = \cos^2 x$. Determine the Taylor polynomial $T_{2,0}(y)$ of second order of f.

Solution: First we compute the first and second derivative of f:

$$f'(x) = -2\cos x \sin x,$$
$$f''(x) = 2(\sin^2 x - \cos^2 x).$$

Hence, $f(0) = 1$, $f'(0) = 0$ and $f''(0) = -2$ which implies that

$$T_{2,0}(y) = f(0) + f'(0)y + \frac{1}{2}f''(0)y^2 = 1 - y^2.$$

Exercise 8 (10 points).

Solve the integral

$$\int_1^4 e^{\sqrt{x}}\,\mathrm{d}x.$$

Solution: In substituting $\sqrt{x} = y$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$ we get

$$\begin{aligned}\int e^{\sqrt{x}}\,\mathrm{d}x &= \int 2ye^{y}\mathrm{d}y \\ &= 2ye^{y} - \int 2e^{y}\mathrm{d}y \\ &= 2ye^{y} - 2e^{y} \\ &= (2\sqrt{x}-2)e^{\sqrt{x}}.\end{aligned}$$

It follows that

$$\int_1^4 e^{\sqrt{x}}\,\mathrm{d}x = (2\sqrt{x}-2)e^{\sqrt{x}}\Big|_1^4 = 2e^2.$$

Exercise 9 (10 points).

Solve the double integral

$$\iint_D x^2 \cos(xy)\, \mathrm{d}(x, y),$$

where

$$D = \{(x, y)^T \in \mathbb{R}^2 : 0 \leqslant x \leqslant \sqrt{\pi}, 0 \leqslant y \leqslant x\}.$$

Solution:

$$\begin{aligned}
\iint_D x^2 \cos(xy)\, \mathrm{d}(x, y) &= \int_0^{\sqrt{\pi}} \int_0^{x} x^2 \cos(xy)\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_0^{\sqrt{\pi}} x^2 \left(\frac{\sin(xy)}{x} \Big|_0^x \right) \mathrm{d}x \\
&= \int_0^{\sqrt{\pi}} x \sin(x^2)\, \mathrm{d}x \\
&= -\frac{1}{2} \cos(x^2) \Big|_0^{\sqrt{\pi}} \\
&= -\frac{1}{2} (\cos \pi - \cos 0) \\
&= 1.
\end{aligned}$$